30^{th} United States of America Mathematical Olympiad

Part I 9 a.m. – 12 noon

May 1, 2001

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

3. Let a, b, and c be nonnegative real numbers such that
$$a^2 + b^2 + c^2 + abc = 4.$$
Prove that
$$0 \leq ab + bc + ca - abc \leq 2.$$

30^{th} United States of America Mathematical Olympiad

Part II 1 p.m. – 4 p.m.

May 1, 2001

4. Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

5. Let S be a set of integers (not necessarily positive) such that

   (A) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$;

   (B) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

   Prove that S is the set of all integers.

6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.