Rocket City Math League 2006-2007

Discovery Solutions — Round 3

1. Answer: 10/3

$$\left(\log_5 4+\log_{125}16\right)\left(\log_2 5\right)$$
$$=\left(2\log_5 2+\frac{4}{3}\log_5 2\right)\left(\log_2 5\right)$$
$$=\left(\frac{10}{3}\log_5 2\right)\left(\log_2 5\right)$$
$$=\frac{10}{3}.$$

2. Answer: 2

$$\det\left(\begin{bmatrix}5 & 7\\ 4 & 6\end{bmatrix}\right)=\begin{vmatrix}5 & 7\\ 4 & 6\end{vmatrix}=5\cdot 6-4\cdot 7=30-28=2.$$

3. Answer: 5

$$f\left(g(5)\right)=f\left(\frac{1}{2}\cdot 5-\frac{1}{2}\right)=f(2)=2\cdot 2+1=5.$$

Also, notice that f and g are inverse functions. The composition of any two functions with a value in both domains yields that value, or $f\left(g(x)\right)=x$. Thus, $f\left(g(5)\right)=5$.

4. Answer: $\frac{\sqrt{6}}{2}$

Let $\sin 15^\circ+\cos 15^\circ=x$.

Square the equation: $\left(\sin 15^\circ+\cos 15^\circ\right)^2=x^2$

Expand: $\sin^2 15^\circ+2\left(\sin 15^\circ\right)\left(\cos 15^\circ\right)+\cos^2 15^\circ=x^2$

Simplify: $1+\sin 30^\circ=1+\frac{1}{2}=x^2$

$$\frac{3}{2}=x^2$$

Because $\sin 15^\circ$ and $\cos 15^\circ$ are both positive, we will take the positive root: $x=\frac{\sqrt{6}}{2}$.

5. Answer: 0

$$x^4+3x^3+3x^2+3x+2=0$$

Use synthetic division to factor the equation:

$$x^4+3x^3+3x^2+3x+2=(x+1)(x+2)(x+i)(x-i)$$

From this, we can see that the sum of the imaginary solutions is 0.

6. Answer: $8\sqrt{3}$

Because the area of the circle is 8π, the radius of the circle will be $2\sqrt{2}$

Created are four figures with three vertices. One figure has three concave arcs, one has two concave arcs and one convex arc, one has one concave arc and two convex arcs, and the last has three convex arcs. From this, connect the three vertices of each figure. We will now pair the figures together: three concave arcs with three convex arcs and the other two together. We will "cut" the protruding parts of convex arcs and add them to the "missing" portions of the concave portions. Resulting will be four equilateral triangles with side length $2\sqrt{2}$.

The area of the four figures will be the area of four equilateral triangles:

$$4\cdot\frac{\left(2\sqrt{2}\right)^2\sqrt{3}}{4}=8\sqrt{3}.$$

7. Answer: 1365

The question is equivalent to asking how many combinations of a, b, c, d, and e are possible if a+b+c+d+e=11 and a, b, c, d, and e are nonnegative integers (including 0) since we can add 1 to a, b, c, d, and e to form the original problem (since 0 was not allowed in the original problem). Now, the question is equivalent to asking how many ways to sort 11 sticks (+) and 4 boundaries (_), where the boundaries break the sticks into groups representing a, b, c, d, and e, i.e. +++_+++_+++_+_+ would represent a=3, b=3, c=3, d=1, and e=1. Thus, the total number of combinations is $\binom{15}{4} = \frac{15!}{11!4!} = 1365$.

8. Answer: $\frac{576}{25}$

The first graph can be written $\frac{x}{3} = \cos\theta$ and $\frac{y}{4} = \sin\theta$ then $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2\theta + \sin^2\theta = 1$. The second graph gives $\theta = \pm\frac{\pi}{4}$ which gives two lines which, in rectangular form, are the lines $y = \pm x$. Substituting y = x into the first equation gives $25x^2 = 144$ or $x = \pm\frac{12}{5}$ so, $y = \pm\frac{12}{5}$. The points of intersection are: $\left(\frac{12}{5},\frac{12}{5}\right)$, $\left(-\frac{12}{5},\frac{12}{5}\right)$, $\left(-\frac{12}{5},-\frac{12}{5}\right)$, and $\left(\frac{12}{5},-\frac{12}{5}\right)$. The figure is a square with side length $\frac{24}{5}$ and area $\frac{576}{25}$.

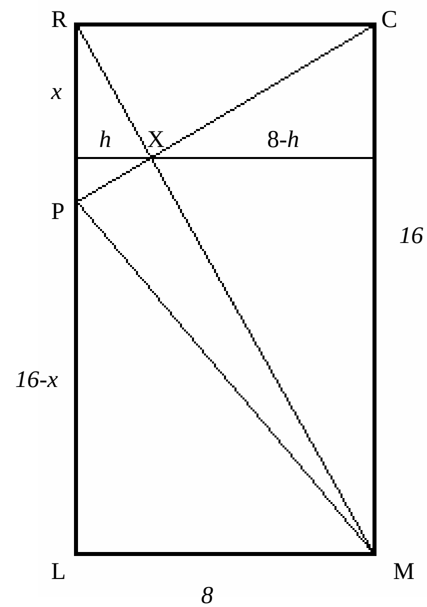
9. Answer: 2857

Applying the principle of inclusion-exclusion, to find out how many numbers are divisible by 2, 3, or 7, we first add the number of numbers divisible by 2, the number of numbers divisible by 3, and the number of numbers divisible by 7, then subtract the number of numbers divisible by 6, the number of numbers divisible by 14, and the number of numbers divisible by 21and then add the number of numbers divisible by 42. Thus, the number of parking space numbers that were erased is

$$\left\lfloor\frac{9999}{2}\right\rfloor + \left\lfloor\frac{9999}{3}\right\rfloor + \left\lfloor\frac{9999}{7}\right\rfloor - \left\lfloor\frac{9999}{6}\right\rfloor - \left\lfloor\frac{9999}{14}\right\rfloor - \left\lfloor\frac{9999}{21}\right\rfloor + \left\lfloor\frac{9999}{42}\right\rfloor$$
$$= 7142$$

The number of parking space numbers that were not erased is 2857.

10. Answer: $96\sqrt{2} - 128$



Let RP = x.

$[PLM] = \frac{1}{2}\cdot 8\cdot(16-x)$ and $[PMX] = [RPM] - [RXP] = \frac{1}{2}\cdot 8x - \frac{1}{2}\cdot hx$

Because $[PLM] = 2[PMX]$,

$$\frac{1}{2}\cdot 8\cdot(16-x) = 8x - hx$$
$$4(16-x) = 8x - hx$$

We will use similar triangles to solve for h.

$$\Delta RXP \approx \Delta MXC$$
$$\frac{h}{x} = \frac{8-h}{16}$$
$$16h = 8x - hx$$
$$16h + hx = 8x$$
$$h = \frac{8x}{x+16}$$

Plug in h in the first equation:

$$4(16-x) = 8x - \frac{8x^2}{x+16}$$

$$4(256 - x^2) = 8x(x+16) - 8x^2$$
$$256 - x^2 = 2x^2 + 32x - 2x^2$$
$$0 = x^2 + 32x - 256$$

Thus, the length of *RP* is $16(\sqrt{2}-1)$.

The area of $\Delta RXP = \frac{1}{2}xh$.

$$h = \frac{8x}{x+16} = \frac{8(16(\sqrt{2}-1))}{16\sqrt{2}-16+16} = \frac{8(16(\sqrt{2}-1))}{16\sqrt{2}} = \frac{8(\sqrt{2}-1)}{\sqrt{2}} = 8-4\sqrt{2}$$

Finally, $[RXP] = \frac{1}{2}xh = \frac{1}{2}(16(\sqrt{2}-1))(8-4\sqrt{2}) = 96\sqrt{2} - 128$.

11. Answer: $\frac{1}{9}$

Since $\log_{14} x$ and $\log_{14} y$ are both integers, the four numbers must be a number in the form $2^a 7^b$, where a and b are non-negative integers. Thus, the possible values of page numbers are 2 (1), 4 (2), 8 (3), 16 (4), 32 (5), 7 (-1), 49 (-2), 1 (0), 14 (0), 28 (1), where the number in parentheses is the power of 2 minus the power of 7 for each number. Therefore, if we just consider the numbers in parentheses, they have to add up to 0 for the four chosen numbers. The only possible combinations for this to happen are

1, 1, -2, 0
1, -1, 0, 0
2, -2, 0, 0
1, 2, -1, -2
3, -1, -2, 0

There are 2 ways to choose the first combination, 2 for the second, 1 for the second, 1 for the third, 2 for the fourth, and 2 for the fifth. Therefore, there are 9 total possible combinations, and the probability that they both chose the same four numbers is 1/9.

12. Answer: $\frac{81}{4}\pi$

Drop perpendicular from Q to OB and call the intersection Point S.
The length of the radius can be written as:

(1) $r^2 = PQ^2 + QS^2 = 7^2 + QS^2$

Length QS can be written as:

(2) $QS^2 = QR^2 - SR^2 = 6^2 - SR^2$

Substituting (2) into (1):

(3) $r^2 = 7^2 + 6^2 - SR^2$

From (3), we know that *SR* is the limiting factor in the length of *r*. *SR* will need to be maximized in order to minimize the length of the radius. This can be accomplished if Point R is Point B.

After establishing that OR has length *r*, we can write (2):

(2.1) $QS^2 = QR^2 - SR^2 = 6^2 - (r-7)^2$

Again, substitute (2.1) into (1),

$$r^2 = 7^2 + 6^2 - (r-7)^2 = 49 + 36 - (r^2 - 14r + 49)$$
$$r^2 = 49 + 36 - r^2 + 14r - 49$$
$$r^2 = 36 - r^2 + 14r$$
$$2r^2 - 14r - 36 = 0$$
$$r^2 - 7r - 18 = 0$$
$$(r-9)(r+2) = 0 \Rightarrow r = 9$$

The radius is 9, so the area of the quarter circle will be:

$$\frac{1}{4}9^2\pi = \frac{81}{4}\pi.$$