Rocket City Math League 2005-2006
Apollo Solutions Round 3

1. Answer: **1**

$x^3 - 3x - 2 = 0$

To find the roots of the equation, factor synthetically.

$x^3 - 3x - 2 = (x+1)^2(x-2) = 0$

Sum = -1 + 2 = 1

2. Answer: **-3/2**

The equation y = 2x + 6 is in slope-intercept form so the slope = 2.

In the second equation , 3x + Ay = 0, so Ay = -3x, y = $\frac{-3}{A}$ x, slope = $\frac{-3}{A}$.

Parallel lines have equal slopes, so $\frac{-3}{A}$ must equal 2. Therefore, A = -3/2.

3. Answer: **7**

There are 33 Thrae days in three Thrae weeks. Let x be the cost of the more expensive medicine. Thus:

$429 = 33[x + (x-1)]$

$13 = 2x - 1$

$14 = 2x$ Thus x =7

4. Answer: **45**

There are $3 \cdot 5 \cdot 3 = 45$ different ways Zaphod can order.

5. Answer: 9^{9^9} or $9^{(9^9)}$

The largest number that can be formed is 9^{9^9}

6. Answer: **1**

$$\left|\frac{(1+2i)}{(2+i)}\right| = \left|\frac{(1+2i)(2-i)}{4+1}\right| = \left|\frac{2-i+4i+2}{5}\right| = \left|\frac{4+3i}{5}\right|$$

$$= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{5^2}{5^2}} = 1$$

7. Answer: **50 + 50i**

$$\sum_{k=0}^{99} (-1)^k (100-k)(i)^{k+1}$$

k = 0	1(100)(i)	= 100i
k = 1	(-1)(99)(-1)	= 99
k = 2	(1)(98)(-i)	= -98i
k = 3	(-1)(97)(1)	= -97
k = 4	(1)(96)(i)	= 96i
… …		…
k = 96	…	= 4i
k = 97	…	= 3
k = 98	…	= -2i
k = 99	…	= -1

Σ = (100i + 99 – 98i – 97) + (96i + 95 – 94i – 93) + … + (4i + 3 – 2i – 1)
= (2i + 2) + (2i + 2) + … + (2i + 2)
= 25(2i + 2)
= 50i + 50

8. Answer: **10**

$\log_a b + \log_b a = \frac{5}{2}$. Let $\log_a b$ = x, then $\log_b a = \frac{1}{\log_a b} = \frac{1}{x}$.

Solving for x: $x + \frac{1}{x} = \frac{5}{2}$, $\frac{x^2+1}{x} = \frac{5}{2}$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

x = -1/2, 2

Now, we solve for a and b and a

$\log_a b = 2$

$a^2 = b$ (ab = 64, so b = 64/a)

$\frac{64^2}{b^2} = b$

$64^2 = b^3$

$(2^6)^2 = b^3$

b = 16

a = 4

Average = $\frac{16+4}{2} = 10$

9. Answer: **28**
One option, of course, is to count them. Let x = the largest possible value (7), and list corresponding values for y and z. The following solution would be helpful for more complicated versions of the same problem:
The solutions for x, y, and z have to be greater than zero, so simplify the equation by substituting:

a = x + 1
b = y + 1
c = z + 1
Thus a + b + c = 6, where a, b, and c are greater than or equal to zero

To solve this equation, we will use six points (•) to illustrate one unit, and two divider (|) to differentiate between a, b, and c. For example: • | • • • | • • is equivalent to a = 1, b = 3, and c = 2; and | • • • • | • • is equivalent to a = 0, b = 4, and c = 2, etc.

There are eight places (6 points and 2 dividers) such that the number of ways to choose values for a b and c is $\frac{8!}{6!2!} = 28$

10. Answer: **237**

First, find the page number of A:

Numbers	# of Digits in Page Number	Total # of Digits for all Numbers
1-9	1	1(9) = 9
10-99	2	2(90) = 180
100-999	3	3(N) = 3N

The sum of the numbers of digits for pages 1 – 99 is equal to 189, which is less than 249. Now, we must find out how many more pages he will have read, so that the sum of the number of digits is equal to 249.

189 + 3N = 249
3N = 60
N = 20, therefore Homer has read 20 3-digit pages, or he has read up to page 119.

If page 119 is halfway through the rocketry text book, then the total number of pages is equal to (119 x 2) – 1 = 237.

11. Answer: $\underline{6+4\sqrt{2}}$

Given: $-3+\sqrt{113+72\sqrt{2}}$

Start with the radical:

$\sqrt{113+72\sqrt{2}} = \sqrt{a}+\sqrt{b}$

$113+72\sqrt{2} = a+b+2\sqrt{ab}$

$a+b=113$

$72\sqrt{2} = 2\sqrt{ab}$

$a = 32$

$b = 81$

$113+72\sqrt{2} = 4\sqrt{2}+9$

Therefore: $-3+\sqrt{113+72\sqrt{2}} = -3+4\sqrt{2}+9 = 6+4\sqrt{2}$

12. Answer: **240**
Total number of ways – total number of ways in which they cannot sit
Total number of ways = 6! = 720
Let A_1, A_2, B_1, B_2, C, C_2 represent each person
cases: (1) A_1 A_2, (2) B_1 B_2, (3) C_1 C_2, (4) A_1 A_2 and B_1 B_2, (5) A_1 A_2 and C_1 C_2, (6) B_1 B_2 and C_1 C_2 only, (7) A_1 A_2 and B_1 B_2 and C_1 C_2.

Consider cases (1), (2), and (3)
Think of A_1 A_2 as one unit, x:
x • 4 • 2 • 1 • 1 8 ways
4 • x • 2 • 1 • 1 8 ways
4 • 2 • x • 2 • 1 16 ways
4 • 2 • 1 • x • 1 8 ways
4 • 2 • 1 • 1 • x 8 ways
total: 48 • 2 ways (can have A_2 A_1) = 96

Now we have the case where two couple sit together: Cases (4), (5), and (6)
So we can set A_1 A_2 = x and B_1 B_2 = y:
x • y • **…** Becomes Case (7)
x • 2 • y • 1 2 ways
x • • • y Becomes Case (7)
2 • x • y • 1 2 ways
2 • x • 1 • y 2 ways
• • x • y Becomes Case (7)
Total: 6 • 2 (switch x and y) • 4 (different orientations: A_1 A_2 B_1 B_2, A_1 A_2 B_2 B_1, A_2 A_1 B_1 B_2, A2 A_1 B_2 B_1) = 48
For Case (7), let A_1 A_2 = x, B_1 B_2 = y, C_1 C_2 = z:
x • y • z
3 • 2 • 1= 6 ways to order x, y, and z;
Total: 6 • 8 (different orientations: A_1 A_2 B_1 B_2 C_1 C_2, A_1 A_2 B_1 B_2 C_2 C_1, etc.)

So the total number of ways to sit: 720 – 3 • 96 – 3 • 48 – 48 = 240