The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions
Presented by The Akamai Foundation

4th Annual American Mathematics Contest 10

AMC 10 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 26, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. Douglas Faires
Department of Mathematics
Youngstown State University
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. (C) We have

$$\frac{2-4+6-8+10-12+14}{3-6+9-12+15-18+21} = \frac{2(1-2+3-4+5-6+7)}{3(1-2+3-4+5-6+7)} = \frac{2}{3}.$$

2. (D) The cost of each day's pills is $546/14 = 39$ dollars. If x denotes the cost of one green pill, then $x + (x-1) = 39$, so $x = 20$.

3. (B) Let n be the smallest of the even integers. Since

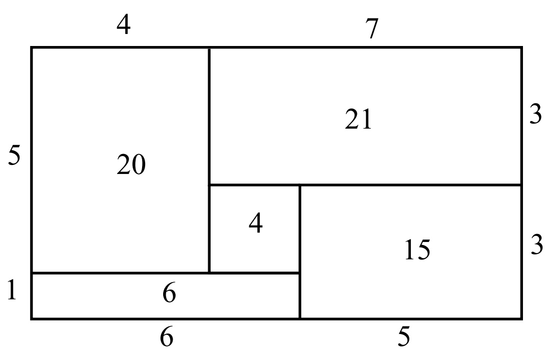
$$1+3+5+7+9+11+13+15 = 64,$$

we have

$$60 = n + (n+2) + (n+4) + (n+6) + (n+8) = 5n + 20, \quad \text{so} \quad n = 8.$$

4. (A) To minimize the cost, Rose should place the most expensive flowers in the smallest region, the next most expensive in the second smallest, etc. The areas of the regions are shown in the figure, so the minimal total cost, in dollars, is

$$(3)(4) + (2.5)(6) + (2)(15) + (1.5)(20) + (1)(21) = 108.$$



5. (C) The area of the lawn is

$$90 \cdot 150 = 13{,}500 \text{ ft}^2.$$

Moe cuts about two square feet for each foot he pushes the mower forward, so he cuts $2(5000) = 10{,}000$ ft^2 per hour. Therefore, it takes about $\frac{13{,}500}{10{,}000} = 1.35$ hours.

6. (D) The height, length, and diagonal are in the ratio $3:4:5$. The length of the diagonal is 27, so the horizontal length is

$$\frac{4}{5}(27) = 21.6 \text{ inches}.$$

7. (B) The first three values in the sum are 1, the next five are 2, the next seven are 3, and the final one is 4 for a total of

$$3 \cdot 1 + 5 \cdot 2 + 7 \cdot 3 + 1 \cdot 4 = 38.$$

8. (B) Let the sequence be denoted $a, ar, ar^2, ar^3, \ldots$, with $ar = 2$ and $ar^3 = 6$. Then $r^2 = 3$ and $r = \sqrt{3}$ or $r = -\sqrt{3}$. Therefore $a = \frac{2\sqrt{3}}{3}$ or $a = -\frac{2\sqrt{3}}{3}$.

9. (B) Write all the terms with the common base 5. Then

$$5^{-4} = 25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}} = \frac{5^{48/x}}{5^{26/x} \cdot 5^{34/x}} = 5^{(48-26-34)/x} = 5^{-12/x}.$$

It follows that $-\frac{12}{x} = -4$, so $x = 3$.

OR

First write 25 as 5^2. Raising both sides to the x power gives

$$5^{-4x} = \frac{5^{48}}{5^{26}5^{34}} = 5^{48-26-34} = 5^{-12}.$$

So $-4x = -12$ and $x = 3$.

10. (C) In the old scheme 26×10^4 different plates could be constructed. In the new scheme $26^3 \times 10^3$ different plates can be constructed. There are

$$\frac{26^3 \times 10^3}{26 \times 10^4} = \frac{26^2}{10}$$

times as many possible plates with the new scheme.

11. (A) The two lines have equations

$$y - 15 = 3(x - 10) \quad \text{and} \quad y - 15 = 5(x - 10).$$

The x-intercepts, obtained by setting $y = 0$ in the respective equations, are 5 and 7. The distance between the points $(5, 0)$ and $(7, 0)$ is 2.

12. (C) Denote the original portions for Al, Betty, and Clare as a, b, and c, respectively. Then

$$a + b + c = 1000 \quad \text{and} \quad a - 100 + 2(b + c) = 1500.$$

Substituting $b + c = 1000 - a$ in the second equation, we have

$$a - 100 + 2(1000 - a) = 1500.$$

This yields $a = 400$, which is Al's original portion.

Note that although we know that $b + c = 600$, we have no way of determining either b or c.

13. (E) Let $y = \clubsuit(x)$. Since $x \le 99$, we have $y \le 18$. Thus if $\clubsuit(y) = 3$, then $y = 3$ or $y = 12$. The 3 values of x for which $\clubsuit(x) = 3$ are 12, 21, and 30, and the 7 values of x for which $\clubsuit(x) = 12$ are 39, 48, 57, 66, 75, 84, and 93. There are 10 values in all.

14. (D) Since a must be divisible by 5, and $3^8 \cdot 5^2$ is divisible by 5^2, but not by 5^3, we have $b \le 2$. If $b = 1$, then

$$a^b = (3^8 5^2)^1 = (164,025)^1 \quad \text{and} \quad a + b = 164,026.$$

If $b = 2$, then

$$a^b = (3^4 5)^2 = 405^2 \quad \text{so} \quad a + b = 407,$$

which is the smallest value.

15. (E) In the first round $100 - 64 = 36$ players are eliminated, one per match. In the second round there are 32 matches, in the third 16, then 8, 4, 2, and 1. The total number of matches is:

$$36 + 32 + 16 + 8 + 4 + 2 + 1 = 99.$$

Note that 99 is divisible by 11, but 99 does not satisfy any of the other conditions given as answer choices.

OR

In each match, precisely one player is eliminated. Since there were 100 players in the tournament and all but one is eliminated, there must be 99 matches.

16. (E) Let m denote the number of main courses needed to meet the requirement. Then the number of dinners available is $3 \cdot m \cdot 2m = 6m^2$. Thus m^2 must be at least $365/6 \approx 61$. Since $7^2 = 49 < 61 < 64 = 8^2$, 8 main courses is enough, but 7 is not.

17. (B) Let r be the radius of the sphere and cone, and let h be the height of the cone. Then the conditions of the problem imply that

$$\frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^2 h, \quad \text{so } h = 3r.$$

Therefore, the ratio of h to r is $3:1$.

18. (D) Among five consecutive odd numbers, at least one is divisible by 3 and exactly one is divisible by 5, so the product is always divisible by 15. The cases $n = 2$, $n = 10$, and $n = 12$ demonstrate that no larger common divisor is possible, since 15 is the greatest common divisor of $3\cdot5\cdot7\cdot9\cdot11$, $11\cdot13\cdot15\cdot17\cdot19$, and $13 \cdot 15 \cdot 17 \cdot 19 \cdot 21$.

19. (E) The area of the larger semicircle is

$$\frac{1}{2}\pi\,(2)^2 = 2\pi.$$

The region deleted from the larger semicircle consists of five congruent sectors and two equilateral triangles. The area of each of the sectors is

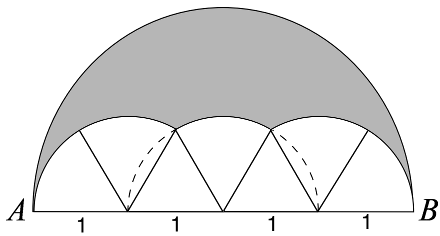
$$\frac{1}{6}\pi (1)^2 = \frac{\pi}{6}$$

and the area of each triangle is

$$\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4},$$

so the area of the shaded region is

$$2\pi - 5 \cdot \frac{\pi}{6} - 2 \cdot \frac{\sqrt{3}}{4} = \frac{7}{6}\pi - \frac{\sqrt{3}}{2}.$$

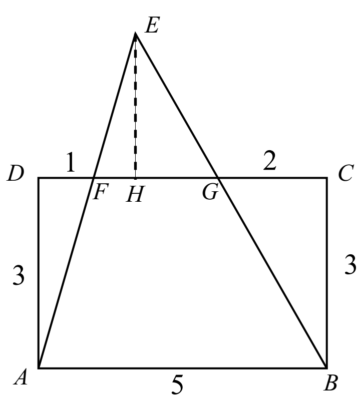


20. (D) Let H be the foot of the perpendicular from E to $\overline{DC}$. Since $CD = AB = 5$, $FG = 2$, and $\triangle FEG$ is similar to $\triangle AEB$, we have

$$\frac{EH}{EH+3} = \frac{2}{5}, \quad \text{so} \quad 5EH = 2EH + 6,$$

and $EH = 2$. Hence

$$\text{Area}(\triangle AEB) = \frac{1}{2}(2+3) \cdot 5 = \frac{25}{2}.$$



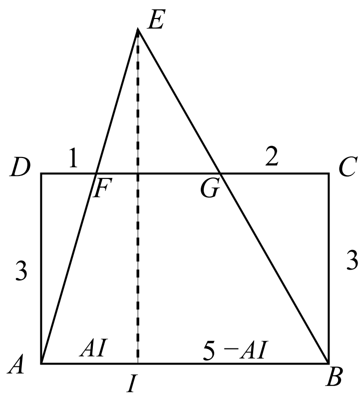
OR

Let I be the foot of the perpendicular from E to $\overline{AB}$. Since

$$\triangle EIA \text{ is similar to } \triangle ADF \quad \text{and} \quad \triangle EIB \text{ is similar to } \triangle BCG,$$

we have

$$\frac{AI}{EI} = \frac{1}{3} \quad \text{and} \quad \frac{5 - AI}{EI} = \frac{2}{3}.$$



Adding gives $5/EI = 1$, so $EI = 5$. The area of the triangle is $\frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

21. (C) The beads will all be red at the end of the third draw precisely when two green beads are chosen in the three draws. If the first bead drawn is green, then there will be one green and three red beads in the bag before the second draw. So the probability that green beads are drawn in the first two draws is

$$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

The probability that a green bead is chosen, then a red bead, and then a green bead, is

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{32}.$$

Finally, the probability that a red bead is chosen then two green beads is

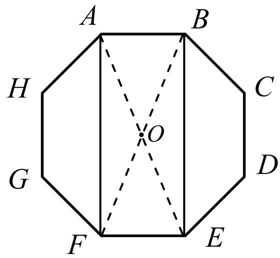
$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{16}.$$

The sum of these probabilities is

$$\frac{1}{8} + \frac{3}{32} + \frac{1}{16} = \frac{9}{32}.$$

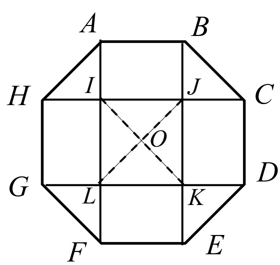
22. (B) In any twelve-hour period, there are 12 half-hour chimes and $1+2+3+\ldots+12 = 78$ on-the-hour chimes. Hence, a twelve-hour period results in 90 chimes. Dividing 2003 by 90 yields a quotient of $22.2\overline{5}$. Therefore the 2003^{rd} chime will occur a little more than 11 days later, on March 9.

23. (D) Let O be the intersection of the diagonals of $ABEF$. Since the octagon is regular, $\triangle AOB$ has area $1/8$. Since O is the midpoint of $\overline{AE}$, $\triangle OAB$ and $\triangle BOE$ have the same area. Thus $\triangle ABE$ has area $1/4$, so $ABEF$ has area $1/2$.



OR

Let O be the intersection of the diagonals of the square $IJKL$. Rectangles $ABJI$, $JCDK$, $KEFL$, and $LGHI$ are congruent. Also $IJ = AB = AH$, so the right isosceles triangles $\triangle AIH$ and $\triangle JOI$ are congruent. By symmetry, the area in the center square $IJKL$ is the sum of the areas of $\triangle AIH$, $\triangle CJB$, $\triangle EKD$, and $\triangle GLF$. Thus the area of rectangle $ABEF$ is half the area of the octagon.



24. (E) Since the difference of the first two terms is $-2y$, the third and fourth terms of the sequence must be $x - 3y$ and $x - 5y$. Thus

$$x - 3y = xy \quad \text{and} \quad x - 5y = \frac{x}{y},$$

so $xy - 5y^2 = x$. Combining these equations we obtain

$$(x - 3y) - 5y^2 = x \quad \text{and, therefore,} \quad -3y - 5y^2 = 0.$$

Since y cannot be 0, we have $y = -3/5$, and it follows that $x = -9/8$. The fifth term in the sequence is $x - 7y = 123/40$.

25. (B) A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So a four-digit number $ab23$ is divisible by 3 if and only if the two-digit number ab leaves a remainder of 1 when divided by 3. There are 90 two-digit numbers, of which $90/3 = 30$ leave a remainder of 1 when divided by 3.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Canada/USA Mathpath & Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
Consortium for Mathematics and its Applications
Institute for Operations Research and the Management Sciences
Kappa Mu Epsilon
Mu Alpha Theta
National Association of Mathematicians
National Council of Teachers of Mathematics
Pi Mu Epsilon
School Science and Mathematics Association
Society of Actuaries