THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

26th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME II)

Wednesday, April 2, 2008

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY and WEDNESDAY, April 29-30, 2008.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Let $N = 100^2 + 99^2 - 98^2 - 97^2 + 96^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$, where the additions and subtractions alternate in pairs. Find the remainder when N is divided by 1000.

2. Rudolph bikes at a constant rate and stops for a five-minute break at the end of every mile. Jennifer bikes at a constant rate which is three-quarters the rate that Rudolph bikes, but Jennifer takes a five-minute break at the end of every two miles. Jennifer and Rudolph begin biking at the same time and arrive at the 50-mile mark at exactly the same time. How many minutes has it taken them?

3. A block of cheese in the shape of a rectangular solid measures 10 cm by 13 cm by 14 cm. Ten slices are cut from the cheese. Each slice has a width of 1 cm and is cut parallel to one face of the cheese. The individual slices are not necessarily parallel to each other. What is the maximum possible volume in cubic cm of the remaining block of cheese after ten slices have been cut off?

4. There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$ and r unique integers a_k $(1 \le k \le r)$ with each a_k either 1 or -1 such that

$$a_1 3^{n_1} + a_2 3^{n_2} + \cdots + a_r 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

5. In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37^\circ$, $\angle D = 53^\circ$, and M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length MN.

6. The sequence $\{a_n\}$ is defined by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \ge 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, b_1 = 3, \text{ and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \ge 2.$$

Find $\dfrac{b_{32}}{a_{32}}$.

7. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

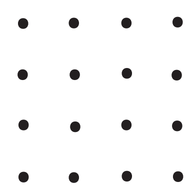
8. Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer.

9. A particle is located on the coordinate plane at $(5, 0)$. Define a *move* for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.

10. The diagram below shows a 4×4 rectangular array of points, each of which is 1 unit away from its nearest neighbors.



Define a *growing path* to be a sequence of distinct points of the array with the property that the distance between consecutive points of the sequence is strictly increasing. Let m be the maximum possible number of points in a growing path, and let r be the number of growing paths consisting of exactly m points. Find mr.

11. In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to circle P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where m, n, and k are positive integers and k is the product of distinct primes. Find $m + nk$.

12. There are two distinguishable flagpoles, and there are 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. Let N be the number of distinguishable arrangements using all of the flags in which each flagpole has at least one flag and no two green flags on either pole are adjacent. Find the remainder when N is divided by 1000 .

13. A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let $S = \{\frac{1}{z} \mid z \in R\}$. Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

14. Let a and b be positive real numbers with $a \geq b$. Let ρ be the maximum possible value of $\frac{a}{b}$ for which the system of equations

$$a^2 + y^2 = b^2 + x^2 = (a - x)^2 + (b - y)^2$$

has a solution (x, y) satisfying $0 \leq x < a$ and $0 \leq y < b$. Then ρ^2 can be expressed as a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Find the largest integer n satisfying the following conditions:
(i) n^2 can be expressed as the difference of two consecutive cubes;
(ii) $2n + 79$ is a perfect square.

Your Exam Manager will receive a copy of the 2008 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2008 USAMO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday and Wednesday, April 29-30, 2008. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.shtml
MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm