THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

26th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Wednesday, April 2, 2008

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 100)

   Reordering the sum shows that

   $N = (100^2 - 98^2) + (99^2 - 97^2) + (96^2 - 94^2) + \cdots + (4^2 - 2^2) + (3^2 - 1^2),$

   which equals

   $$\begin{aligned} & 2 \cdot 198 + 2 \cdot 196 + 2 \cdot 190 + 2 \cdot 188 + \cdots + 2 \cdot 6 + 2 \cdot 4 \\ = & \, 4(99 + 98 + 95 + 94 + \cdots + 3 + 2) \\ = & \, 4(99 + 95 + 91 + \cdots + 3) + 4(98 + 94 + 90 + \cdots + 2) \\ = & \, 4\left[\frac{25(99+3)}{2}\right] + 4\left[\frac{25(98+2)}{2}\right] \\ = & \, 100 \cdot (51 + 50) \\ = & \, 10100, \end{aligned}$$

   and the required remainder is 100.

2. (Answer: 620)

   Suppose Rudolph bikes at r miles per minute. He takes 49 five-minute breaks in reaching the 50-mile mark, so his total time in minutes is $\frac{50}{r} + 49 \cdot 5 = \frac{50}{r} + 245$. Jennifer bikes at $\frac{3}{4}r$ miles per minute and takes 24 five-minute breaks in reaching the 50-mile mark, so her total time in minutes is $\frac{50}{0.75r} + 24 \cdot 5 = \frac{200}{3r} + 120$. Setting these two times equal gives $\frac{50}{3r} = 125$, and hence $r = \frac{2}{15}$. This yields a total time of $\frac{50}{2/15} + 245 = 620$ minutes.

3. (Answer: 729)

   Let a, b, and c be the dimensions of the cheese after ten slices have been cut off, giving a volume of abc cu cm. Because each slice shortens one of the dimensions of the cheese by 1 cm, $a+b+c = (10+13+14) - 10 = 27$. By the Arithmetic-Geometric Mean Inequality, the product of a set of positive numbers with a given sum is greatest when the numbers are equal, so the remaining cheese has maximum volume when $a = b = c = 9$. The volume is then $9^3 = 729$ cu cm.

   Query: What happens if $a + b + c$ is not divisible by 3?

   OR

   The volume of the remaining cheese is greatest when it forms a cube. This can be accomplished by taking one slice from the 10 cm dimension, 4 slices from the 13 cm dimension, and 5 slices from the 14 cm dimension for a total of $1 + 4 + 5 = 10$ slices. The remaining cheese is then 9 cm by 9 cm by 9 cm for a volume of 729 cu cm.

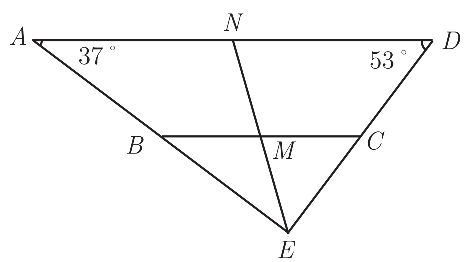
4. (Answer: 021)

Every positive integer has a unique base-3 representation, which for 2008 is 2202101_3. Note that $2 \cdot 3^k = (3 + (-1)) \cdot 3^k = 3^{k+1} + (-1) \cdot 3^k$, so that

$$\begin{aligned}
&2202101_3 \\
&= 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^0 \\
&= (3^7 + (-1) \cdot 3^6) + (3^6 + (-1) \cdot 3^5) + (3^4 + (-1) \cdot 3^3) + 1 \cdot 3^2 + 1 \cdot 3^0 \\
&= 1 \cdot 3^7 + (-1) \cdot 3^5 + 1 \cdot 3^4 + (-1) \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^0,
\end{aligned}$$

and $n_1 + n_2 + \cdots + n_r = 7 + 5 + 4 + 3 + 2 + 0 = 21$.

5. (Answer: 504)



Extend leg $\overline{AB}$ past B and leg $\overline{CD}$ past C, and let E be the point of intersection of these extensions. Then because $\frac{BM}{AN} = \frac{CM}{DN}$, line MN must pass through point E. But $\angle A = 37°$ and $\angle D = 53°$ implies that $\angle AED = 90°$. Thus $\triangle EDA$ is a right triangle with median $\overline{EN}$, and $\triangle EBC$ is a right triangle with median $\overline{EM}$. The median to the hypotenuse in any right triangle is half the hypotenuse, so $EN = \frac{2008}{2} = 1004$, $EM = \frac{1000}{2} = 500$, and $MN = EN - EM = 504$.

6. (Answer: 561)

Observe that if $x_n = x_{n-1} + \dfrac{x_{n-1}^2}{x_{n-2}}$ and $y_n = \dfrac{x_n}{x_{n-1}}$, then $y_n = 1 + y_{n-1}$, so

$$\frac{x_n}{x_0} = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_1}{x_0} = y_1 y_2 \cdots y_n = y_1(y_1 + 1) \cdots (y_1 + n - 1).$$

In particular, for the first sequence, $y_1 = \frac{a_1}{a_0} = 1$, and so $a_n = n!$. Similarly, for the second sequence, $y_1 = \frac{b_1}{b_0} = 3$, and so $b_n = \frac{1}{2}(n+2)!$. The required ratio is then $\frac{34!/2}{32!} = 17 \cdot 33 = 561$.

7. (Answer: 753)

   Because the equation is cubic and there is no x^2 term, the sum of the roots is 0; that is, $r + s + t = 0$. Therefore,

   $$(r+s)^3 + (s+t)^3 + (t+r)^3 = (-t)^3 + (-r)^3 + (-s)^3 = -(r^3 + s^3 + t^3).$$

   Because r is a root, $8r^3 + 1001r + 2008 = 0$, and similarly for s and t. Therefore, $8(r^3 + s^3 + t^3) + 1001(r + s + t) + 3 \cdot 2008 = 0$, and

   $$r^3 + s^3 + t^3 = \frac{1001(r+s+t) + 3 \cdot 2008}{-8} = \frac{3 \cdot 2008}{-8} = -753,$$

   and hence $(r+s)^3 + (s+t)^3 + (t+r)^3 = -(r^3 + s^3 + t^3) = 753$.

8. (Answer: 251)

   The product-to-sum formula for $2\cos(k^2a)\sin(ka)$ yields $\sin(k^2a + ka) - \sin(k^2a - ka)$, which equals $\sin(k(k+1)a) - \sin((k-1)ka)$. Thus the given sum becomes

   $$\sin(2 \cdot 1a) - \sin(0) + \sin(3 \cdot 2a) - \sin(2 \cdot 1a) + \sin(4 \cdot 3a) - \sin(3 \cdot 2a) + \cdots \\ + \sin(n(n+1)a) - \sin((n-1)na).$$

   This is a telescoping sum, which simplifies to $\sin(n(n+1)a) - \sin(0) = \sin(n(n+1)a)$. But $\sin(x)$ is an integer only when x is an integer multiple of $\pi/2$, so $n(n+1)$ must be an integer multiple of $1004 = 2^2 \cdot 251$. Thus either n or $n+1$ is a multiple of 251, because 251 is prime. The smallest such n is 250, but $250 \cdot 251$ is not a multiple of 1004. The next smallest such n is 251, and $251 \cdot 252$ is a multiple of 1004. Hence the smallest such n is 251.

9. (Answer: 019)

   Let the coordinate plane be the complex plane, and let z_k be the complex number that represents the position of the particle after k moves. Multiplying a complex number by cis θ corresponds to a rotation of θ about the origin, and adding 10 to a complex number corresponds to a horizontal translation of 10 units to the right. Thus $z_0 = 5$, and $z_{k+1} = \omega z_k + 10$, where $\omega = \text{cis}\ (\pi/4)$, for $k \geq 0$. Then

   $$\begin{aligned} z_1 &= 5\omega + 10, \\ z_2 &= \omega(5\omega + 10) + 10 = 5\omega^2 + 10\omega + 10, \\ z_3 &= \omega(5\omega^2 + 10\omega + 10) + 10 = 5\omega^3 + 10\omega^2 + 10\omega + 10, \end{aligned}$$

   and, in general,

   $$z_k = 5\omega^k + 10(\omega^{k-1} + \omega^{k-2} + \cdots + 1).$$

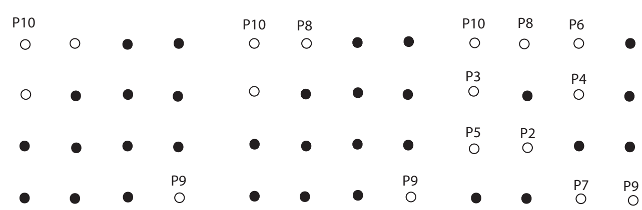
In particular, $z_{150} = 5\omega^{150} + 10(\omega^{149} + \omega^{148} + \cdots + 1)$. Note that $\omega^8 = 1$ and $\omega^{k+4} = \text{cis }((k+4)\pi/4) = \text{cis }(k\pi/4 + \pi) = -\text{cis }(k\pi/4) = -\omega^k$. Applying the second equality repeatedly shows that $z_{150} = 5\omega^{150} + 10(\omega^{149} + \omega^{148} + \cdots + 1) = 5\omega^6 + 10(-\omega + (-1) + \omega^3 + \omega^2 + \omega + 1) = 5\omega^6 + 10(\omega^3 + \omega^2)$. The last expression equals

$$\begin{aligned} &5\text{cis }(3\pi/2) + 10(\text{cis }(3\pi/4) + \text{cis }(\pi/2)) \\ =&5(-i) + 10\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i + i\right) \\ =&-5\sqrt{2} + (5\sqrt{2} + 5)i \end{aligned}$$

Thus $(p, q) = (-5\sqrt{2}, 5\sqrt{2}+5)$, and $|p|+|q| = 10\sqrt{2}+5$. The required value is therefore the greatest integer less than or equal to $10 \cdot 1.414 + 5 = 19.14$, which is 19.

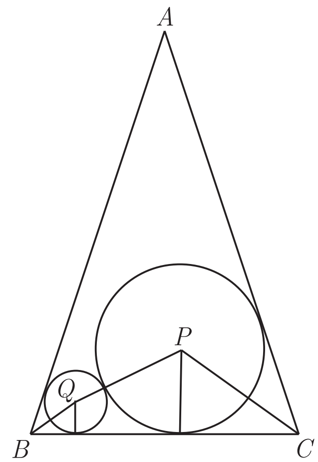
10. (Answer: 240)

The following argument shows that $m = 10$ and $r = 24$. Note that the square of any distance between distinct points in the array has the form $a^2 + b^2$ for some integers a and b in the set $\{0, 1, 2, 3\}$ with a and b not both equal to 0. There are 9 such possible values, namely, 1, 2, 4, 5, 8, 9, 10, 13, and 18. Hence a growing path can consist of at most 10 points. It remains to show that 10 points can be so chosen, and that there are 24 such paths. Let the points of these paths be labeled $P_1, P_2, \ldots, P_{10}$. First, P_9P_{10} must be $\sqrt{18}$; that is, P_9 and P_{10} must be a pair of opposite corners in the array. There are 4 equivalent ways to label P_9 and P_{10}. Without loss of generality, assume that P_9 and P_{10} are the bottom right corner and upper left corner, respectively. Then there are 2 symmetrical positions for P_8, both neighboring P_{10}. (See the left-hand diagram below.) Assume that P_8 is beside P_{10}, as shown in the middle diagram below.



Note that the bottom left corner cannot be P_7, because otherwise $P_6 = P_9$ or $P_6 = P_{10}$. The positions of points P_7, P_6, P_5, P_4, P_3, and P_2 are then fixed. (See the right-hand diagram above.) Finally, there are 3 possible positions for P_1. Hence $r = 4 \cdot 2 \cdot 3 = 24$, and $mr = 240$.

11. (Answer: 254)



Let q be the radius of circle Q. The perpendicular from A to $\overline{BC}$ has length $4\sqrt{25^2-7^2}=4\cdot 24$, and so $\sin B=\sin C=\frac{24}{25}$. Thus

$$\tan\frac{C}{2}=\tan\frac{B}{2}=\frac{\sin B}{1+\cos B}=\frac{3}{4}.$$

Place the figure in a Cartesian coordinate system with $B=(0,0)$, $C=(56,0)$, and $A=(28,96)$. Note that circles P and Q are both tangent to $\overline{BC}$, and their centers P and Q lie on the angle bisectors of angles C and B, respectively. Thus $P=(56-4\cdot 16/3,16)$ and $Q=(4q/3,q)$. Also, $PQ=q+16$, and so

$$PQ^2=\left(\frac{4(q+16)}{3}-56\right)^2+(q-16)^2=(q+16)^2,$$

which yields $(4q-104)^2=576q$, and thus $q^2-88q+676=0$. The roots of this equation are $44\pm 6\sqrt{35}$, but $44+6\sqrt{35}$ is impossible because it exceeds the base of $\triangle ABC$. Hence $q=44-6\sqrt{35}$, and the requested number is $44+6\cdot 35=254$.

12. (Answer: 310)

For convenience, label the flagpoles 1 and 2, and denote by G_n the number of flag arrangements in which flagpole 1 has n green flags. If either flagpole contains all the green flags, then it must also contain at least 8 blue flags to act as separators. The flagpole with no green flags must therefore contain either 1 or 2 blue flags. Hence, $G_0=G_9=11$, because there are 10 possible positions in which to place 1 blue flag on the flagpole with all the green flags (otherwise, the flagpole with no green flags contains 2 blue flags). If $0 < n < 9$, then a flagpole that has n green flags must contain at least $n-1$ blue flags, and the other flagpole has $9-n$ green flags and must contain at least $8-n$ blue flags. Thus in any flag arrangement where each flagpole contains at least one green flag, 7 of the blue flags have fixed positions relative to the green flags, and there are 3 remaining blue flags that can be freely distributed. If flagpole 1 has n green flags, then there are $n+1$ possible positions in which 1 blue flag can be placed on that flagpole, and there are $10-n$ possible positions in which 1 blue flag can be placed on flagpole 2. Thus, for $0 < n < 9$, after placing the green flags and 7 of the blue flags, there remain 3 blue flags with 11 distinguishable locations. This is a standard problem of distributing 3 indistinguishable objects among 11 distinguishable bins. Because 11 bins require 10 separators to divide them, the problem is equivalent to choosing 3 locations out of 13 (the 3 flags and the 10 separators). Thus the number of ways to place the blue flags is $\binom{13}{3}$. The desired number of flag arrangements is then

$$N = \sum_{i=0}^{9} G_i = 2 \cdot 11 + 8\binom{13}{3} = 2310,$$

and the required remainder is 310.

13. (Answer: 029)

One pair of vertices lies at $\frac{1}{2} \pm \frac{1}{2\sqrt{3}}i$. Express points on the line segment determined by these two vertices in the form $z = \frac{1}{2} + yi$, where y is real and $|y| \le \frac{1}{2\sqrt{3}}$. Reciprocals of points on this line segment are then of the form $\dfrac{\frac{1}{2} - yi}{\frac{1}{4} + y^2}$ with $|y| \le \frac{1}{2\sqrt{3}}$. Because

$$\left|\frac{1}{z} - 1\right|^2 = \left|\frac{\frac{1}{2} - yi}{\frac{1}{4} + y^2} - \frac{\frac{1}{4} + y^2}{\frac{1}{4} + y^2}\right|^2 = \frac{(\frac{1}{4} - y^2)^2 + y^2}{(\frac{1}{4} + y^2)^2} = \frac{(\frac{1}{4} + y^2)^2}{(\frac{1}{4} + y^2)^2} = 1,$$

the curve traced by the reciprocals of complex numbers on this line segment is an arc of a circle centered at 1 with radius 1, running from $\frac{3}{2} - \frac{\sqrt{3}}{2}i$ to $\frac{3}{2} + \frac{\sqrt{3}}{2}i$. The region enclosed by this arc and the lines from the origin to the endpoints can be partitioned into a 120-degree sector of the disk centered at 1 with radius 1, together with two triangles, each of base 1 and height $\frac{\sqrt{3}}{2}$. Thus it has area $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$. Multiply by six to find that the total area is $2\pi + 3\sqrt{3} = 2\pi + \sqrt{27}$. Thus $a + b = 29$.

Query: The above argument shows that the reciprocals of the points on a line not through the origin fall on a circle through the origin. What happens to a line through the origin?

OR

Because pairs of parallel sides are one unit apart, one side of the hexagon lies along the line $x = \text{Re } z = \frac{1}{2}$. If $w = u + vi = \frac{1}{z}$, where z is on this line, then

$$\frac{1}{2}\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) = \frac{1}{2},$$

which is equivalent to $w+\overline{w} = w\overline{w}$. Hence $u^2+v^2 = 2u$, or $(u-1)^2+v^2 = 1$, which is the equation of a circle of radius 1 centered at 1. Because $z = 1$ is mapped to $w = 1$, it follows that $\text{Re } z > \frac{1}{2}$ is mapped to the interior of this circle. By symmetry, the five remaining half-planes whose union produces R are mapped to corresponding disks. Thus the points that are in the image of the half-plane $\text{Re } z > \frac{1}{2}$ but none of the other five half-planes belong to the disk $(u-1)^2+v^2 < 1$ and the region bounded by the two rays $v = \pm\frac{u}{\sqrt{3}}$. The resulting set can be partitioned into a circular sector with radius 1 and central angle $\frac{2\pi}{3}$ and two isosceles triangles with equal sides of length 1 and vertex angle $\frac{2\pi}{3}$. There are six such nonoverlapping congruent figures forming S. It follows that the area of S is $6(\frac{\pi}{3} + \frac{\sqrt{3}}{2}) = 2\pi + \sqrt{27}$. Thus $a + b = 29$.

14. (Answer: 007)

Let $ABCD$ be a rectangle such that $AB = CD = a$, and $BC = DA = b$. Let E and F be points on the sides $\overline{AB}$ and $\overline{BC}$ respectively, such that $AE = x$ and $CF = y$. Solving the given system of equations is thus equivalent to requiring that $\triangle DEF$ be equilateral. Let $\angle ADE = \alpha$. Then $\angle FDC = 30^\circ - \alpha$, $\tan\alpha = \frac{x}{b}$, and $\angle EDF = 60^\circ$. Thus

$$\frac{y}{a} = \tan(30^\circ - \alpha) = \frac{\frac{1}{\sqrt{3}} - \frac{x}{b}}{1 + \frac{1}{\sqrt{3}} \cdot \frac{x}{b}} = \frac{b - x\sqrt{3}}{b\sqrt{3} + x}. \tag{1}$$

Squaring and adding 1 yields $\dfrac{4(x^2 + b^2)}{(b\sqrt{3} + x)^2} = \dfrac{y^2 + a^2}{a^2} = \dfrac{x^2 + b^2}{a^2}$. Thus $4a^2 = (b\sqrt{3} + x)^2$, and $x \geq 0$ implies that $x = 2a - b\sqrt{3}$, which is a positive real number because $a \geq b$. Equation (1) shows that $y \geq 0$ if and only if $b - x\sqrt{3} = b - (2a - b\sqrt{3})\sqrt{3} = 4b - 2a\sqrt{3} \geq 0$. It follows that $\frac{a}{b} \leq \frac{2}{\sqrt{3}}$, and so $\rho = \frac{2}{\sqrt{3}}$. Hence $\rho^2 = \frac{4}{3}$, and $m + n = 7$. This value of ρ is achieved when $a = 2, b = \sqrt{3}, x = 1$, and $y = 0$.

15. (Answer: 181)

Let m be an integer such that $(m + 1)^3 - m^3 = n^2$, which implies that $3(2m+1)^2 = (2n-1)(2n+1)$. Because $2n-1$ and $2n+1$ are consecutive odd integers, they are relatively prime. Thus 3 can only divide one of $2n - 1$ and $2n + 1$. Therefore either $2n - 1 = 3k^2$ and $2n + 1 = j^2$, or $2n - 1 = k^2$ and $2n + 1 = 3j^2$. The first case implies that $j^2 - 3k^2 = 2$, which can be shown to be impossible by examining the equation modulo 3. The second case implies that $4n = 3j^2 + k^2$ and $3j^2 - k^2 = 2$. Let $k = 2a+1$ for some integer a. Then $3j^2 = k^2+2 = (2a+1)^2+2$. Therefore $4n = (2a+1)^2 + (2a+1)^2 + 2 = 8a^2 + 8a + 4$, or $n = 2a^2 + 2a + 1$. Furthermore, because $2n + 79 = d^2$ for some integer d, then $2n + 79 = d^2 = 2(2a^2 + 2a + 1) + 79 = 4a^2 + 4a + 81$. This equation is equivalent to $80 = d^2 - (4a^2 + 4a + 1) = d^2 - (2a+1)^2 = (d-2a-1)(d+2a+1)$. Both factors on the right side are of the same parity, so they both must be even. Then the two factors on the right are either $(2, 40)$, $(4, 20)$, or $(8, 10)$, and $(d, a) = (21, 9)$, $(12, \frac{7}{2})$, $(9, 0)$. The first solution gives $n = \frac{441-79}{2} = 181$, and the last solution gives $n = \frac{81-79}{2} = 1$. Thus the largest such n is 181 (with $m = 104$).

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