The Mathematical Association of America
American Mathematics Competitions

23rd Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 22, 2005

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:
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1. (Answer: 013)

   The conditions of the problem imply that $\binom{n}{6} = 6\binom{n}{3}$, so $\frac{n!}{6!(n-6)!} = 6 \cdot \frac{n!}{3!(n-3)!}$. Then $\frac{(n-3)!}{(n-6)!} = 6!$, so $(n-3)(n-4)(n-5) = 720 = 10 \cdot 9 \cdot 8$. Thus $n = 13$ is a solution, and because $(n-3)(n-4)(n-5)$ is increasing for $n \geq 5$, conclude that 13 is the only solution for $n \geq 5$.

2. (Answer: 079)

   The probability that the first bag contains one of each of the three types of rolls is
   $(9/9)(6/8)(3/7) = 9/28$. The probability that the second bag will then contain one of each is $(6/6)(4/5)(2/4) = 2/5$. If the first two bags have a complete selection, then the last bag must too. Thus the probability that all three breakfasts have a complete selection is $(9/28)(2/5) = 9/70$, and $m + n = 9 + 70 = 79$.

3. (Answer: 802)

   Let a be the first term and r the ratio of the original series, and let $S = 2005$. Then $\frac{a}{1-r} = S$ and $\frac{a^2}{1-r^2} = 10S$. Factor to obtain $10S = \left(\frac{a}{1-r}\right)\left(\frac{a}{1+r}\right) = S \cdot \frac{a}{1+r}$. Then $10 = \frac{a}{1+r}$ and $S = \frac{a}{1-r}$ imply that $S(1-r) = 10(1+r)$, so $r = \frac{S-10}{S+10} = 1995/2015 = 399/403$, and $m + n = 802$.

4. (Answer: 435)

   Note that $10^{10} = 2^{10}5^{10}$, so it has $11 \cdot 11 = 121$ divisors. Similarly, $15^7 = 3^7 \cdot 5^7$, so it has $8 \cdot 8 = 64$ divisors, and $18^{11} = 2^{11}3^{22}$, so it has $12 \cdot 23 = 276$ divisors. There are 8 divisors of both 10^{10} and 15^7, namely those numbers that are divisors of 5^7; there are 11 divisors of both 10^{10} and 18^{11}, namely those numbers that are divisors of 2^{10}; and there are 8 divisors of both 15^7 and 18^{11}, namely those numbers that are divisors of 3^7. There is only one divisor of all three. Therefore, the Inclusion-Exclusion Principle implies that the number of divisors of at least one of the numbers is $(121 + 64 + 276) - (8 + 11 + 8) + 1 = 435$.

5. (Answer: 054)

   Let $x = \log_a b$. Because $\log_b a = 1/\log_a b$, the given equation can be written as $x + (6/x) = 5$, and because $x \neq 0$, this is equivalent to $x^2 - 5x + 6 = 0$, whose solutions are 2 and 3. If $2 = x = \log_a b$, then $a^2 = b$. Now $44^2 = 1936$ and $45^2 = 2025$, so there are $44 - 1 = 43$ ordered pairs (a, b) such that $a^2 = b$ and

a and b satisfy the given conditions. If $3 = x = \log_a b$, then $a^3 = b$. Because $12^3 = 1728$ and $13^3 = 2197$, there are $12 - 1 = 11$ ordered pairs (a, b) such that $a^3 = b$ and a and b satisfy the given conditions. Thus there are $43 + 11 = 54$ of the requested ordered pairs.

6. (Answer: 392)

Note that, after the restacking, all the cards from pile B occupy even-numbered positions and their order is reversed. Similarly, all the cards from pile A will be placed in odd-numbered positions, and their order is also reversed. A card in position i for $1 \le i \le n$ will be moved to position $2(n-i)+1$ in the restacking, and, for $n < i \le 2n$, the card will be moved to position $2(2n-i)+2$. For a card to remain in the 131st position, it must be in pile A. Then $131 = 2(n-131)+1$, and $2n = 392$. Note that the stack is magical because cards number 131 and 262 retain their original positions.

7. (Answer: 125)

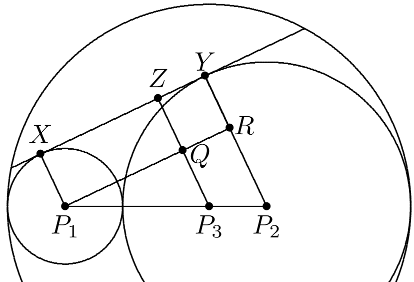
Let $y = \sqrt[16]{5}$. Then

$$\begin{aligned} x = \frac{4}{(y^8+1)(y^4+1)(y^2+1)(y+1)} &= \frac{4(y-1)}{(y^8+1)(y^4+1)(y^2+1)(y+1)(y-1)} \\ &= \frac{4(y-1)}{y^{16}-1} = \frac{4(y-1)}{5-1} = y-1. \end{aligned}$$

Thus $(x+1)^{48} = y^{48} = 5^3 = 125$.

8. (Answer: 405)

The radius of $\mathcal{C}_3$ is 14. Let P_1, P_2, and P_3 be the centers of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively. Draw perpendiculars from P_1, P_2, and P_3 to the external tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$ intersecting it at X, Y, and Z, respectively, so that $\overline{P_1X}$, $\overline{P_2Y}$, $\overline{P_3Z}$ are parallel, with $P_1X = 4$ and $P_2Y = 10$. From P_1, draw a line parallel to $\overline{XY}$ intersecting $\overline{P_3Z}$ and $\overline{P_2Y}$ at Q and R, respectively. Note that P_1XYR is a rectangle and that right triangles P_1P_2R and P_1P_3Q are similar. Then $P_3Z = P_3Q + QZ = (10/14) \cdot 6 + 4 = 58/7$. Because Z is the midpoint of the chord, the chord's length is $2\sqrt{14^2 - (58/7)^2} = 8\sqrt{390}/7$, and $m + n + p = 8 + 390 + 7 = 405$.



9. (Answer: 250)

Note that

$$\begin{aligned}(\sin t + i\cos t)^n &= \left[\cos\left(\frac{\pi}{2} - t\right) + i\sin\left(\frac{\pi}{2} - t\right)\right]^n \\ &= \cos n\left(\frac{\pi}{2} - t\right) + i\sin n\left(\frac{\pi}{2} - t\right) \\ &= \cos\left(\frac{n\pi}{2} - nt\right) + i\sin\left(\frac{n\pi}{2} - nt\right),\end{aligned}$$

and that $\sin nt + i\cos nt = \cos\left(\frac{\pi}{2} - nt\right) + i\sin\left(\frac{\pi}{2} - nt\right)$. Thus the given condition is equivalent to

$$\cos\left(\frac{n\pi}{2} - nt\right) = \cos\left(\frac{\pi}{2} - nt\right) \quad \text{and} \quad \sin\left(\frac{n\pi}{2} - nt\right) = \sin\left(\frac{\pi}{2} - nt\right).$$

In general, $\cos\alpha = \cos\beta$ and $\sin\alpha = \sin\beta$ if and only if $\alpha - \beta = 2\pi k$. Thus

$$\frac{n\pi}{2} - nt - \frac{\pi}{2} + nt = 2\pi k,$$

which yields $n = 4k + 1$. Because $1 \le n \le 1000$, conclude that $0 \le k \le 249$, so there are 250 values of n that satisfy the given conditions.

OR

Observe that

$$\begin{aligned}&(\sin t + i\cos t)^n = [i(\cos t - i\sin t)]^n = i^n(\cos nt - i\sin nt), \quad \text{and that} \\ &\sin nt + i\cos nt = i(\cos nt - i\sin nt).\end{aligned}$$

Thus the given equation is equivalent to $i^n(\cos nt - i\sin nt) = i(\cos nt - i\sin nt)$. This is true for all real t when $i^n = i$. Thus n must be 1 more than a multiple of 4, so there are 250 values of n that satisfy the given conditions.

10. (Answer: 011)

Without loss of generality, let the edges of $\mathcal{O}$ have length 1. Note that $\mathcal{O}$ can be formed by adjoining two square pyramids at their bases. Consider an altitude from a vertex of one of these pyramids to the square base. This altitude is also a leg of a right triangle whose other leg joins the center of the square and a vertex of the square, and whose hypotenuse is an edge of $\mathcal{O}$. The length of the altitude is therefore $\sqrt{1^2 - \left(\sqrt{2}/2\right)^2} = \sqrt{2}/2$, and so the volume of $\mathcal{O}$ is $2 \cdot (1/3)(1^2)(\sqrt{2}/2) = \sqrt{2}/3$. To find the volume of $\mathcal{C}$, consider a triangle, one of whose vertices P is a vertex of $\mathcal{O}$ not on the square and whose other two vertices, Q and R, are midpoints of opposite sides of the square. The line segment that joins the centers of the faces containing $\overline{PQ}$ and $\overline{PR}$, respectively, is a diagonal of a face of $\mathcal{C}$. Because these centers are two-thirds of the way from P to Q and from P to R, respectively, the length of the face diagonal joining them is two-thirds of QR. But $QR = 1$, so the length of each of the edges of $\mathcal{C}$ is $(2/3)/\sqrt{2} = \sqrt{2}/3$. Hence the volume of $\mathcal{C}$ is $(\sqrt{2}/3)^3 = 2\sqrt{2}/27$. The requested ratio is thus $(\sqrt{2}/3)/(2\sqrt{2}/27) = 9/2$, so $m + n = 11$.

OR

The six vertices of $\mathcal{O}$ are equidistant from its center, and the diagonals that join the three pairs of opposite vertices are mutually perpendicular. Without loss of generality, let the length of each of these three diagonals be 2. It is possible to place a coordinate system so that the coordinates of the vertices of $\mathcal{O}$ are $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$. Because $\mathcal{O}$ is composed of two square pyramids, its volume is $2(1/3)(\sqrt{2})^2 \cdot 1 = 4/3$. The vertices of $\mathcal{C}$ are the centroids of the faces of $\mathcal{O}$, so the coordinates of the vertices of $\mathcal{C}$ are $(\pm 1/3, \pm 1/3, \pm 1/3)$. Thus the length of each of the edges of $\mathcal{C}$ is $2/3$, and the volume of $\mathcal{C}$ is $8/27$. The ratio of the volumes is $(4/3)/(8/27) = 9/2$, so $m + n = 11$.

11. (Answer: 889)

For $1 \le k \le m - 1$, we have $a_{k+1}a_k = a_k a_{k-1} - 3$. Let $b_k = a_k a_{k-1}$ for $1 \le k \le m$. Then $b_1 = a_1 a_0 = 72 \cdot 37 = 3 \cdot 8 \cdot 3 \cdot 37 = 3 \cdot 888$ and $b_{k+1} = b_k - 3$. Hence $b_{889} = 0$ and $b_k > 0$ for $1 \le k \le 888$. Thus $a_{889} = 0$ and $m = 889$.

12. (Answer: 307)

Let G be the midpoint of $\overline{AB}$, let $\alpha = m\angle EOG$, and let $\beta = m\angle FOG$. Then $OG = 450$, $EG = 450\tan\alpha$, $FG = 450\tan\beta$, and $\alpha + \beta = 45^\circ$. Therefore $450(\tan\alpha + \tan\beta) = 400$, so $\tan\alpha + \tan\beta = 8/9$. Notice that $\tan\beta = \tan(45^\circ - \alpha) = \dfrac{1 - \tan\alpha}{1 + \tan\alpha}$. Hence $\tan\alpha + \dfrac{1 - \tan\alpha}{1 + \tan\alpha} = \dfrac{8}{9}$. Simplify to obtain $9\tan^2\alpha - 8\tan\alpha + 1 = 0$, and conclude that $\{\tan\alpha, \tan\beta\} = \{(4 \pm \sqrt{7})/9\}$. Because $BF > AE$, conclude that $EG > FG$, and so $\alpha > \beta$. Then $\tan\alpha = (4 + \sqrt{7})/9$ and $\tan\beta = (4 - \sqrt{7})/9$. Thus

$$\begin{aligned} BF = BG - FG = 450 - 450\tan\beta = 450\left(1 - \frac{4 - \sqrt{7}}{9}\right) &= 450\left(\frac{5 + \sqrt{7}}{9}\right) \\ &= 250 + 50\sqrt{7}, \end{aligned}$$

so $p+q+r=307$.

OR

Draw $\overline{AO}$ and $\overline{BO}$. Then $m\angle OAB = 45^\circ = m\angle EOF$, and $m\angle OEF = m\angle OAB + m\angle AOE = 45^\circ + m\angle AOE = m\angle AOF$. Therefore $\triangle AFO \sim \triangle BOE$, so $\dfrac{AO}{AF} = \dfrac{BE}{BO}$. Let $BF = x$. Then $AF = 900 - x$ and $BE = 400 + x$. Thus

$$\begin{aligned}
&\frac{450\sqrt{2}}{900-x} = \frac{400+x}{450\sqrt{2}}, \quad \text{which yields}\\
&2\cdot 450^2 = 360000 + 500x - x^2, \quad \text{and then}\\
&x^2 - 500x + 45000 = 0.
\end{aligned}$$

Use the Quadratic Formula to obtain $x = 250 \pm 50\sqrt{7}$. Recall that $BF > AE$, and so $x > (900-400)/2 = 250$. Then $BF = x = 250+50\sqrt{7}$, and $p+q+r = 307$.

13. (Answer: 418)

Let $S(x) = P(x) - x - 3$. Because $S(17) = -10$ and $S(24) = -10$,

$$S(x) = -10 + (x-17)(x-24)Q(x)$$

for some polynomial $Q(x)$ with integer coefficients. If n is an integer such that $P(n) = n+3$, then $S(n) = 0$, and $(n-17)(n-24)Q(n) = 10$. Thus the integers $n-17$ and $n-24$ are divisors of 10 that differ by 7. The only such pairs are $(2,-5)$ and $(5,-2)$. This yields $\{n_1, n_2\} = \{19, 22\}$, hence $n_1 \cdot n_2 = 418$. An example of a polynomial that satisfies the conditions of the problem is $P(x) = x - 7 - (x-17)(x-24)$.

14. (Answer: 463)

Let $m\angle BAE = \alpha = m\angle CAD$, and let $\beta = m\angle EAD$. Then

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{(1/2)AB \cdot AD \sin BAD}{(1/2)AD \cdot AC \sin CAD} = \frac{AB}{AC} \cdot \frac{\sin(\alpha+\beta)}{\sin\alpha}.$$

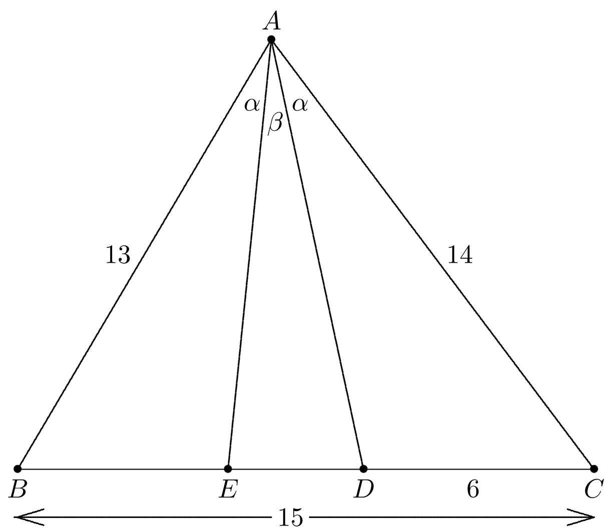
Similarly,

$$\frac{BE}{EC} = \frac{AB}{AC} \cdot \frac{\sin BAE}{\sin CAE} = \frac{AB}{AC} \cdot \frac{\sin\alpha}{\sin(\alpha+\beta)},$$

and so

$$\frac{BE}{EC} = \frac{AB^2 \cdot DC}{AC^2 \cdot BD}.$$

Substituting the given values yields $BE/EC = (13^2 \cdot 6)/(14^2 \cdot 9) = 169/294$. Therefore $BE = (15 \cdot 169)/(169+294) = (3 \cdot 5 \cdot 13^2)/463$. Because none of 3, 5, and 13 divides 463, $q = 463$.



15. (Answer: 169)

Complete the square to obtain $(x+5)^2+(y-12)^2 = 256$ and $(x-5)^2+(y-12)^2 = 16$ for ω_1 and ω_2, respectively. Hence ω_1 is centered at $F_1(-5, 12)$ with radius 16, and ω_2 is centered at $F_2(5, 12)$ with radius 4. Let P be the center of the third circle, and let r be its radius. Then $PF_1 = 16 - r$ and $PF_2 = 4 + r$. Thus P is on the ellipse with foci F_1, F_2 and $PF_1 + PF_2 = 20$. Therefore the coordinates of P satisfy

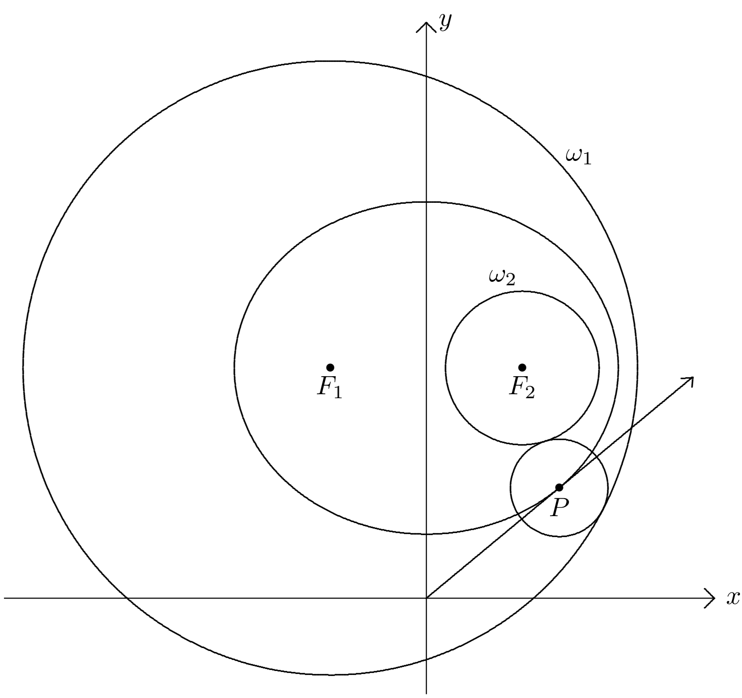
$$\frac{x^2}{100} + \frac{(y-12)^2}{75} = 1,$$

which is equivalent to $3x^2 + 4y^2 - 96y + 576 = 300$. Because P is on the line with equation $y = ax$, conclude that the x-coordinate of P satisifies

$$(3 + 4a^2)x^2 - 96ax + 276 = 0.$$

In order for P to exist, the discriminant of the above quadratic equation must be nonnegative, that is, $(-96a)^2 - 4 \cdot 276 \cdot (4a^2 + 3) \geq 0$. Thus $a^2 \geq 69/100$, so $m^2 = 69/100$, and $p + q = 169$.

Note that a attains its minimum when the line with equation $y = ax$ is tangent to the ellipse.



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