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(AIME)

SOLUTIONS PAMPHLET

Tuesday, March 25, 2003

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 839)

   Note that
$$\frac{((3!)!)!}{3!} = \frac{(6!)!}{6} = \frac{720!}{6} = \frac{720 \cdot 719!}{6} = 120 \cdot 719!.$$
   Because $120 \cdot 719! < 720!$, conclude that n must be less than 720, so the maximum value of n is 719. The requested value of $k + n$ is therefore $120 + 719 = 839$.

2. (Answer: 301)

   The sum of the areas of the green regions is
$$\begin{aligned} &\left[(2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + \cdots + (100^2 - 99^2)\right]\pi \\ =& \left[(2+1) + (4+3) + (6+5) + \cdots + (100+99)\right]\pi \\ =& \frac{1}{2} \cdot 100 \cdot 101\pi. \end{aligned}$$
   Thus the desired ratio is
$$\frac{1}{2} \cdot \frac{100 \cdot 101\pi}{100^2\pi} = \frac{101}{200},$$
   and $m + n = 301$.

3. (Answer: 484)

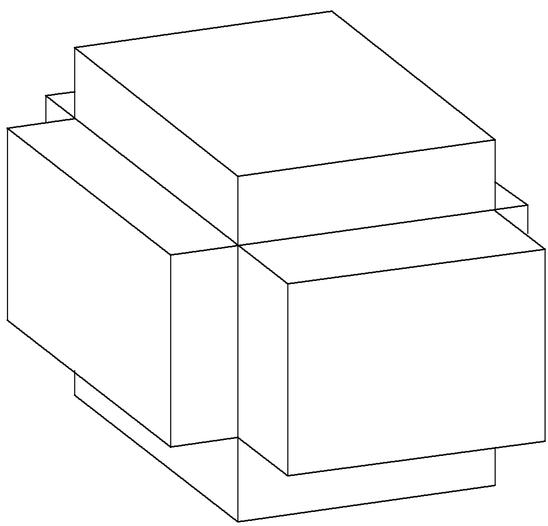
   Since each element x of $\mathcal{S}$ is paired exactly once with every other element in the set, the number of times x contributes to the sum is the number of other elements in the set that are smaller than x. For example, the first number, 8, will contribute four times to the sum because the greater elements of the subsets $\{8, 5\}$, $\{8, 1\}$, $\{8, 3\}$, and $\{8, 2\}$ are all 8. Since the order of listing the elements in the set is not significant, it is helpful to first sort the elements of the set in increasing order. Thus, since $\mathcal{S} = \{1, 2, 3, 5, 8, 13, 21, 34\}$, the sum of the numbers on the list is $0(1)+1(2)+2(3)+3(5)+4(8)+5(13)+6(21)+7(34) = 484$.

4. (Answer: 012)

   Use logarithm properties to obtain $\log(\sin x \cos x) = -1$, and then $\sin x \cos x = 1/10$. Note that
$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \frac{2}{10} = \frac{12}{10}.$$
   Thus
$$2 \log(\sin x + \cos x) = \log \frac{12}{10} = \log 12 - 1,$$
   so
$$\log(\sin x + \cos x) = \frac{1}{2}(\log 12 - 1),$$
   and $n = 12$.

5. (Answer: 505)

First consider the points in the six parallelepipeds projecting 1 unit outward from the original parallelepiped. Two of these six parallelepipeds are 1 by 3 by 4, two are 1 by 3 by 5, and two are 1 by 4 by 5. The sum of their volumes is $2(1\cdot3\cdot4+1\cdot3\cdot5+1\cdot4\cdot5) = 94$. Next consider the points in the twelve quarter-cylinders of radius 1 whose heights are the edges of the original parallelepiped. The sum of their volumes is $4 \cdot \frac{1}{4}\pi \cdot 1^2(3+4+5) = 12\pi$. Finally, consider the points in the eight octants of a sphere of radius 1 at the eight vertices of the original parallelepiped. The sum of their volumes is $8 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi \cdot 1^3 = \frac{4\pi}{3}$. Because the volume of the original parallelepiped is $3 \cdot 4 \cdot 5 = 60$, the requested volume is $60 + 94 + 12\pi + 4\pi/3 = \dfrac{462 + 40\pi}{3}$, so $m + n + p = 462 + 40 + 3 = 505$.



6. (Answer: 348)

The sides of the triangles may be cube edges, face-diagonals of length $\sqrt{2}$, or space-diagonals of length $\sqrt{3}$. A triangle can consist of two adjacent edges and a face-diagonal; three face-diagonals; or an edge, a face-diagonal, and a space-diagonal. The first type of triangle is right with area $1/2$, and there are 24 of them, 4 on each face. The second type of triangle is equilateral with area $\sqrt{3}/2$. There are 8 of these because each of these triangles is uniquely determined by the three vertices adjacent to one of the 8 vertices of the cube. The third type of triangle is right with area $\sqrt{2}/2$. There are 24 of these because there are four space-diagonals and each determines six triangles, one with each cube vertex that is not an endpoint of the diagonal. (Note that there is a total of $\binom{8}{3} = 56$ triangles with the desired vertices, which is consistent with the above results.)

The desired sum is thus $24(1/2) + 8(\sqrt{3}/2) + 24(\sqrt{2}/2) = 12 + 4\sqrt{3} + 12\sqrt{2} = 12 + \sqrt{48} + \sqrt{288}$, and $m + n + p = 348$.

7. (Answer: 380)

Let $AD = CD = a$, let $BD = b$, and let E be the projection of D on $\overline{AC}$. It follows that $a^2 - 15^2 = DE^2 = b^2 - 6^2$, or $a^2 - b^2 = 225 - 36 = 189$. Then $(a+b, a-b) = (189, 1)$, $(63, 3)$, $(27, 7)$, or $(21, 9)$, from which $(a, b) = (95, 94)$, $(33, 30)$, $(17, 10)$, or $(15, 6)$. The last pair is rejected since b must be greater than 6. Because each possible triangle has a perimeter of $2a+30$, it follows that $s = 190 + 66 + 34 + 3 \cdot 30 = 380$.

OR

Let (a_k, b_k) be the possible values for a and b, and let n be the number of possible perimeters of $\triangle ACD$. Then $s = \sum_{k=1}^{n}(30+2a_k) = 30n + \sum_{k=1}^{n}[(a_k+b_k)+(a_k-b_k)]$. But $(a_k + b_k)(a_k - b_k) = a_k^2 - b_k^2 = 189 = 3^3 \cdot 7$ which has 4 pairs of factors. Thus $n = 4$. Therefore the sum of the perimeters of the triangles is $30 \cdot 4$ more than the sum of the divisors of 189, that is, $120 + (1+3+3^2+3^3)(1+7) = 440$. However, this includes the case where $D = E$, the projection of D on $\overline{AC}$, so $s = 440 - 60 = 380$.

8. (Answer: 129)

Let a, $a+d$, $a+2d$, and $\dfrac{(a+2d)^2}{a+d}$ be the terms of the sequence, with a and d positive integers. Then $(a+30)(a+d) = (a+2d)^2$, which yields $3a(10-d) = 2d(2d-15)$. It follows that either $10 - d > 0$ and $2d - 15 > 0$ or $10 - d < 0$ and $2d - 15 < 0$. In the first case, d is 8 or 9, and the second case has no solutions. When $d = 8$, $a = 8/3$, and when $d = 9$, $a = 18$. Thus, the only acceptable sequence is 18, 27, 36, 48, and the sum is 129.

9. (Answer: 615)

For a balanced four-digit integer, the sum of the leftmost two digits must be at least 1 and at most 18. Let $f(n)$ be the number of ways to write n as a sum of two digits where the first is at least 1, and let $g(n)$ be the number of ways to write n as a sum of two digits. For example, $f(3) = 3$, since $3 = 1+2 = 2+1 = 3+0$, and $g(3) = 4$. Then

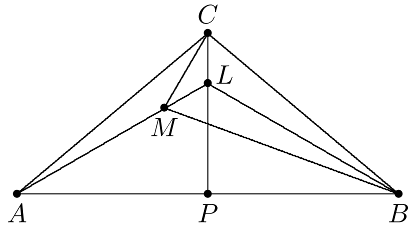
$$f(n) = \begin{cases} n & \text{for} 1 \le n \le 9, \\ 19 - n & \text{for} 10 \le n \le 18, \end{cases} \quad \text{and} \quad g(n) = \begin{cases} n+1 & \text{for} 1 \le n \le 9, \\ 19 - n & \text{for} 10 \le n \le 18. \end{cases}$$

For any balanced four-digit integer whose leftmost and rightmost digit pairs both have sum n, the number of possible leftmost digit pairs is $f(n)$ because the leftmost digit must be at least 1, and the number of possible rightmost digit pairs is $g(n)$. Thus there are $f(n) \cdot g(n)$ four-digit balanced integers whose leftmost and rightmost digit pairs both have sum n. The total number of balanced four-digit integers is then equal to

$$\begin{aligned}\sum_{n=1}^{18} f(n) \cdot g(n) &= \sum_{n=1}^{9} n(n+1) + \sum_{n=10}^{18} (19-n)^2 = \sum_{n=1}^{9}(n^2+n) + \sum_{n=1}^{9} n^2 \\ &= 2\sum_{n=1}^{9} n^2 + \sum_{n=1}^{9} n = 2(1^2+2^2+\cdots+9^2) + (1+2+\cdots+9) \\ &= 615.\end{aligned}$$

10. (Answer: 083)

Let $\overline{CP}$ be the altitude to side $\overline{AB}$. Extend $\overline{AM}$ to meet $\overline{CP}$ at point L, as shown. Since $\angle ACL = 53^\circ$, conclude that $\angle MCL = 30^\circ$. Also, $\angle LMC = \angle MAC + \angle ACM = 30^\circ$. Thus $\triangle MLC$ is isosceles with $LM = LC$ and $\angle MLC = 120^\circ$. Because L is on the perpendicular bisector of $\overline{AB}$, $\angle LBA = \angle LAB = 30^\circ$ and $\angle MLB = 120^\circ$. It follows that $\angle BLC = 120^\circ$. Now consider $\triangle BLM$ and $\triangle BLC$. They share $\overline{BL}$, $ML = LC$, and $\angle MLB = \angle CLB = 120^\circ$. Therefore they are congruent, and $\angle LMB = \angle LCB = 53^\circ$. Hence $\angle CMB = \angle CML + \angle LMB = 30^\circ + 53^\circ = 83^\circ$.



OR

Without loss of generality, assume that $AC = BC = 1$. Apply the Law of Sines in $\triangle AMC$ to obtain

$$\frac{\sin 150^\circ}{1} = \frac{\sin 7^\circ}{CM},$$

from which $CM = 2\sin 7^\circ$. Apply the Law of Cosines in $\triangle BMC$ to obtain $MB^2 = 4\sin^2 7^\circ + 1 - 2 \cdot 2\sin 7^\circ \cdot \cos 83^\circ = 4\sin^2 7^\circ + 1 - 4\sin^2 7^\circ = 1$. Thus $CB = MB$, and $\angle CMB = 83^\circ$.

11. (Answer: 092)

Because $\cos(90^\circ - x) = \sin x$ and $\sin(90^\circ - x) = \cos x$, it suffices to consider x in the interval $0^\circ < x \le 45^\circ$. For such x,

$$\cos^2 x \ge \sin x \cos x \ge \sin^2 x,$$

so the three numbers are not the lengths of the sides of a triangle if and only if

$$\cos^2 x \ge \sin^2 x + \sin x \cos x,$$

which is equivalent to $\cos 2x \ge \frac{1}{2}\sin 2x$, or $\tan 2x \le 2$. Because the tangent function is increasing in the interval $0^\circ \le x \le 45^\circ$, this inequality is equivalent to $x \le \frac{1}{2}(\arctan 2)^\circ$. It follows that

$$p = \frac{\frac{1}{2}(\arctan 2)^\circ}{45^\circ} = \frac{(\arctan 2)^\circ}{90^\circ},$$

so $m + n = 92$.

12. (Answer: 777)

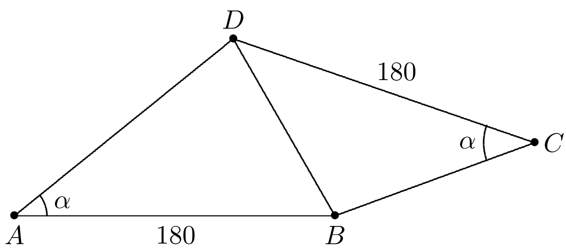
Let $\angle A = \angle C = \alpha$, $AD = x$, and $BC = y$. Apply the Law of Cosines in triangles ABD and CDB to obtain

$$BD^2 = x^2 + 180^2 - 2 \cdot 180x \cos\alpha = y^2 + 180^2 - 2 \cdot 180y \cos\alpha.$$

Because $x \ne y$, this yields

$$\cos\alpha = \frac{x^2 - y^2}{2 \cdot 180(x - y)} = \frac{x + y}{360} = \frac{280}{360} = \frac{7}{9}.$$

Thus $\lfloor 1000 \cos A \rfloor = 777$.



OR

Assume without loss of generality that AD is the greater of AD and BC. Then there is a point P on $\overline{AD}$ with $AP = BC$. Because $\triangle BAP \cong \triangle DCB$, conclude that $BP = BD$, and altitude $\overline{BH}$ of isosceles $\triangle BPD$ bisects $\overline{PD}$. Now $\cos A = AH/180$, and because $AH = AP + (PD/2) = AD - (PD/2)$,

$$AH = \frac{AP + AD}{2} = \frac{BC + AD}{2} = \frac{640 - 2 \cdot 180}{2} = 140.$$

Thus $\cos A = 140/180 = 7/9$, and $\lfloor 1000 \cos A \rfloor = 777$.

13. (Answer: 155)

Since $2003 < 2047 = 2^{11} - 1$, the integers in question have at most 11 digits in base 2. Since the base-2 representation of a positive integer must begin with 1, the number of $(d+1)$-digit numbers with exactly $(k+1)$ 1's is $\binom{d}{k}$. The number of 1's exceeds the number of 0's if and only if $k + 1 > (d+1)/2$, or $k \geq d/2$. Thus the number of integers whose base-2 representation consists of at most 11 digits and that have more 1's than 0's is the sum of the entries in rows 0 through 10 in Pascal's Triangle that are on or to the right of the vertical symmetry line. The sum of all entries in these rows is $1 + 2 + 4 + \cdots + 1024 = 2047$, and the sum of the center elements is $\displaystyle\sum_{i=0}^{5} \binom{2i}{i} = 1 + 2 + 6 + 20 + 70 + 252 = 351$, so the sum of the entries on or to the right of the line is $(2047 + 351)/2 = 1199$. The 44 integers less than 2048 and greater than 2003 all have at least six 1's, because they are all greater than 1984, which is 11111000000 in base 2, so they have all been included in the total. Thus the required number is $1199 - 44 = 1155$, whose remainder upon division by 1000 is 155.

14. (Answer: 127)

To find the smallest n, it is sufficient to consider the case in which the string 251 occurs immediately after the decimal point. To show this, suppose that in the decimal representation of m/n, the string 251 does not occur immediately after the decimal point. Then $m/n = .A251\ldots$, where A represents a block of k digits, $k \geq 1$. This implies that $10^k m/n - A = .251\ldots$, but $10^k m/n - A$, which is between 0 and 1, can then be expressed in the form a/b, where a and b are relatively prime positive integers and $b \leq n$. Now

$$\frac{251}{1000} \leq \frac{m}{n} < \frac{252}{1000}.$$

It follows that $251n \leq 1000m < 252n = 251n + n$. The remainder when $1000m$ is divided by 251 must therefore be less than n, so it is sensible to investigate multiples of 251 that are close to and less than a multiple of 1000. When $m = 1$, $n = 3$ yields $3 \cdot 251 = 753$ as the multiple of 251 that is closest to and less than 1000; but the remainder is greater than 3. When $m = 2$, $n = 7$ yields $3 \cdot 251 + 4 \cdot 251 = 1757$ as the multiple of 251 that is closest to and less than 2000; but the remainder is greater than 7. More generally, $(4m - 1)251$ is less than $1000m$ when $m \leq 62$, and the remainder is $1000m - (4m - 1)251 = 251 - 4m$. The remainder is less than n when $251 - 4m < 4m - 1$, that is, when $m > 31$. Thus the minimum value of m is 32, and the minimum value of n is $4 \cdot 32 - 1 = 127$.

15. (Answer: 289)

Let $AB = c$, $BC = a$, and $CA = b$. Since $a > c$, F is on $\overline{BC}$. Let ℓ be the line passing through A and parallel to $\overline{DF}$, and let ℓ meet $\overline{BD}$, $\overline{BE}$, and $\overline{BC}$ at D', E', and F' respectively. Since $\overline{AF'}$ is parallel to $\overline{DF}$,

$$\frac{m}{n} = \frac{DE}{EF} = \frac{D'E'}{E'F'}.$$

In $\triangle ABF'$, $\overline{BD'}$ is both an altitude and an angle-bisector, so $\triangle ABF'$ is isosceles with $BF' = BA = c$. Hence $AD' = D'F'$, and

$$\frac{AE'}{E'F'} = \frac{AD' + D'E'}{E'F'} = \frac{D'F' + D'E'}{E'F'} = \frac{E'F' + 2D'E'}{E'F'} = 1 + \frac{2m}{n}.$$

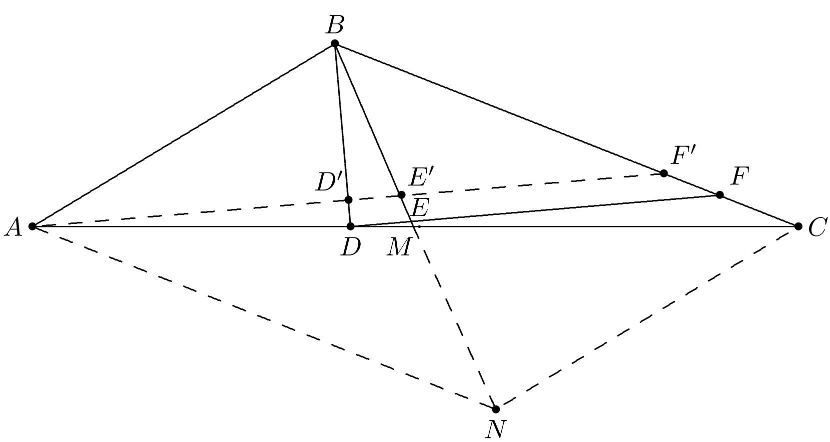
Extend $\overline{BM}$ through M to N so that $BM = MN$, and draw $\overline{AN}$ and $\overline{CN}$. Quadrilateral $ABCN$ is a parallelogram because diagonals $\overline{AC}$ and $\overline{BN}$ bisect each other. Hence $AN = BC = a$ and triangles $AE'N$ and $F'E'B$ are similar. Therefore

$$1 + \frac{2m}{n} = \frac{AE'}{E'F'} = \frac{AN}{F'B} = \frac{a}{c},$$

and

$$\frac{m}{n} = \frac{a - c}{2c} = \frac{507 - 360}{720} = \frac{49}{240},$$

so $m + n = 289$.



OR

Let $AB = c$, $BC = a$, and $CA = b$. Let D' and D'' be the points where the lines parallel to line DF and containing A and C, respectively, intersect $\overrightarrow{BD}$, and let E' and F', be the points where $\overrightarrow{AD'}$ meets $\overline{BM}$ and $\overline{BC}$, respectively. Let G be the point on $\overline{BM}$ so that lines FG and AC are parallel. Note that

$$\frac{c}{a} = \frac{AD}{DC} = \frac{DD'}{DD''} = \frac{FF'}{FC} = \frac{BF - BF'}{BC - BF} = \frac{BF - c}{a - BF},$$

which yields $BF = \dfrac{2ac}{a+c}$.

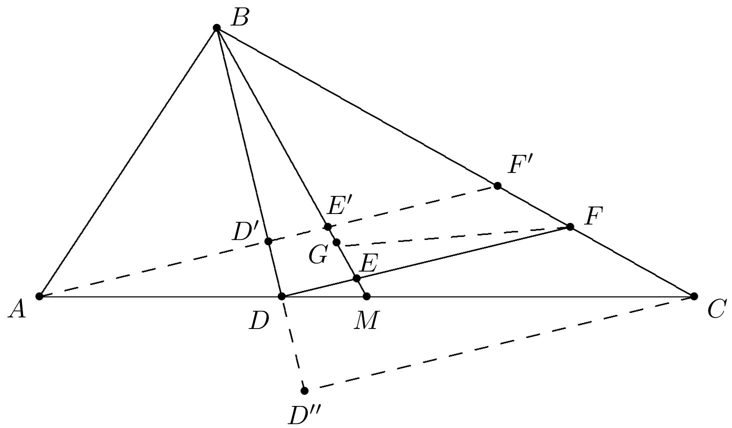
Also, $\dfrac{GF}{MC} = \dfrac{BF}{BC}$, so $GF = \dfrac{b}{2} \cdot \dfrac{BF}{a} = \dfrac{b}{2} \cdot \dfrac{2c}{a+c} = \dfrac{bc}{a+c}$.

But $\dfrac{c}{a} = \dfrac{AD}{DC} = \dfrac{AD}{b - AD}$ yields $AD = \dfrac{bc}{a+c}$.

Therefore $AD = GF$, which implies that $ADFG$ is a parallelogram, so $\overline{AG}$ is parallel to $\overline{DF}$. Thus $G = E'$, and then

$$\frac{DE}{EF} = \frac{DM}{GF} = \frac{AM - AD}{AD} = \frac{\frac{1}{2}(AD + CD) - AD}{AD} = \frac{\frac{1}{2}(CD - AD)}{AD} = \frac{1}{2}\left(\frac{CD}{AD} - 1\right),$$

$$\text{so} \quad \frac{DE}{EF} = \frac{1}{2}\left(\frac{a}{c} - 1\right) = \frac{a - c}{2c}.$$



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