MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS
PRESENTED BY THE AKAMAI FOUNDATION

THE AKAMAI FOUNDATION

20th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

Tuesday, April 9, 2002

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, calculators are not permitted.

4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on FRIDAY and SATURDAY, May 3 & 4, 2002.

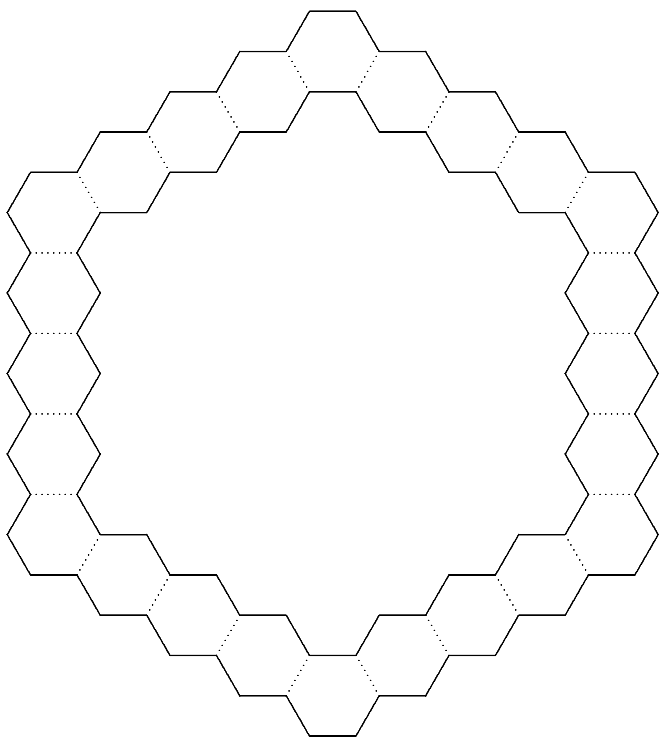
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Given that
   (1) x and y are both integers between 100 and 999, inclusive;
   (2) y is the number formed by reversing the digits of x; and
   (3) $z = |x - y|$.
   How many distinct values of z are possible?

2. Three of the vertices of a cube are $P = (7, 12, 10)$, $Q = (8, 8, 1)$, and $R = (11, 3, 9)$. What is the surface area of the cube?

3. It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

4. Patio blocks that are regular hexagons 1 unit on a side are used to outline a garden by placing the blocks edge to edge with n on each side. The diagram indicates the path of blocks around the garden when $n = 5$.



   If $n = 202$, then the area of the garden enclosed by the path, not including the path itself, is $m(\sqrt{3}/2)$ square units, where m is a positive integer. Find the remainder when m is divided by 1000.

5. Find the sum of all positive integers $a = 2^n 3^m$, where n and m are non-negative integers, for which a^6 is not a divisor of 6^a.

6. Find the integer that is closest to $1000 \sum_{n=3}^{10000} \frac{1}{n^2-4}$.

7. It is known that, for all positive integers k,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \cdots + k^2$ is a multiple of 200.

8. Find the least positive integer k for which the equation $\left\lfloor \frac{2002}{n} \right\rfloor = k$ has no integer solutions for n. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.)

9. Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (*Disjoint sets* are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

10. While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$, where m, n, p and q are positive integers. Find $m+n+p+q$.

11. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $1/8$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.

12. A basketball player has a constant probability of .4 of making any given shot, independent of previous shots. Let a_n be the ratio of shots made to shots attempted after n shots. The probability that $a_{10} = .4$ and $a_n \leq .4$ for all n such that $1 \leq n \leq 9$ is given to be $p^a q^b r/(s^c)$, where p, q, r, and s are primes, and a, b, and c are positive integers. Find $(p+q+r+s)(a+b+c)$.

13. In triangle ABC, point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$, point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$, and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$. It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m+n$.

14. The perimeter of triangle APM is 152, and angle PAM is a right angle. A circle of radius 19 with center O on $\overline{AP}$ is drawn so that it is tangent to $\overline{AM}$ and $\overline{PM}$. Given that $OP = m/n$, where m and n are relatively prime positive integers, find $m+n$.

15. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points, one of which is $(9,6)$, and the product of their radii is 68. The x-axis and the line $y = mx$, where $m > 0$, are tangent to both circles. It is given that m can be written in the form $a\sqrt{b}/c$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a+b+c$.

Your Exam Manager will have a copy of the 2002 AIME Solution Pamphlet.

WRITE TO US:

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2002 USAMO

THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held on Friday and Saturday, May 3 & 4, 2002 at a central site – Cambridge MA. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered as indicated below.

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