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SOLUTIONS PAMPHLET

Tuesday, March 26, 2002

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 059)
The probability that a license plate will have a three-letter palindrome is $\frac{(26^2)}{(26^3)} = \frac{1}{26}$ because there are 26 possibilities for each of the first two letters, and the third letter must be the same as the first. Similarly, the probability that a license plate will have a three-digit palindrome is $(10^2)/(10^3) = 1/10$. The probability that a license plate will have both a three-letter palindrome and a three-digit palindrome is $(1/26)(1/10) = 1/260$. Apply the Inclusion-Exclusion Principle to conclude that the probability that a license plate will have at least one palindrome is
$$\frac{1}{26} + \frac{1}{10} - \frac{1}{260} = \frac{35}{260} = \frac{7}{52}\,.$$
Thus $m + n = 59$.

2. (Answer: 154)
Let r be the radius of each circle, and let l and w be the dimensions of the rectangle with $l > w$. Then $14r = l$. Now consider the equilateral triangle whose vertices are the centers of any three mutually tangent circles. The height of such a triangle is $r\sqrt{3}$, so $w = 2r + 2r\sqrt{3}$. It follows that
$$\frac{l}{w} = \frac{14r}{2r(1+\sqrt{3})} = \frac{7(\sqrt{3}-1)}{2} = \frac{\sqrt{147}-7}{2},$$
so $p + q = 154$.

3. (Answer: 025)
Let Dick's age and Jane's age in n years be $10x + y$ and $10y + x$, respectively. At that time, Dick will be $9(x - y)$ years older than Jane, and the sum of their ages will be $11(x + y)$. Dick's age must always exceed Jane's by a multiple of 9; thus Dick's current age must be 34, 43, 52, 61, 70, 79, 88, or 97. Suppose that Dick is 34, so that the sum of their ages is 59. Their age-sum is therefore always odd, and it is not a multiple of 11 until it reaches 77. This takes $n = \frac{1}{2}(77 - 59) = 9$ years. Every 11 years thereafter, as long as Dick has a two-digit age, their ages will be reversals of each other; Dick's ages at those times are 43, 54, 65, 76, 87, and 98. Similar reasoning applies if Dick's current age is 43, and their age-sum is 68: the next age-sum that has the same parity and is divisible by 11 is 88, when Dick is 53 and $n = 10$. Every 11 years thereafter, until Dick is 97, their ages are reversals of each other — five examples in all. Similarly, there are four examples if Dick's current age is 52, four examples if his current age is 61, three examples if his current age is 70, two examples if his current age is 79, one example if his current age is 88, and none if his current age is 97. The total number of ordered pairs is thus $6 + 5 + 4 + 4 + 3 + 2 + 1 = 25$.

OR

Since Dick must always be older than Jane, in n years Jane may be 26, 27, 28, 29, 34,. . ., 39, 45, . . ., 49, 56, . . ., 59, 67, 68, 69, 78, 79 or 89. Dick's age will be the result of reversing the digits of Jane's age. The total number of ordered pairs is thus $4+6+5+4+3+2+1=25$.

4. (Answer: 840)
Because
$$\frac{1}{k^2+k}=\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1},$$
the series telescopes, that is,
$$\begin{aligned}1/29 &= a_m+a_{m+1}+\cdots+a_{n-1}\\ &= \left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right),\end{aligned}$$
so $(1/m)-(1/n)=1/29$. Since neither m nor n is 0, this is equivalent to $mn+29m-29n=0$, from which we obtain $(m-29)(n+29)=-29^2$, or $(29-m)(29+n)=29^2$. Since 29 is a prime and $29+n>29-m$, it follows that $29-m=1$ and $29+n=29^2$. Thus $m=28$, $n=29^2-29$, and $m+n=29^2-1=30\cdot 28=840$.

5. (Answer: 183)
Each pair $\{A_i,A_j\}$ of the $\binom{12}{2}=66$ pairs of vertices generates three squares, one having diagonal A_iA_j, and the other two having A_iA_j as a side. However, each of the three squares $A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$, and $A_3A_6A_9A_{12}$ is counted six times. The total number of squares is therefore $3\cdot 66-15=183$.

6. (Answer: 012)
Let $p=\log_{225}x=1/\log_x 225$ and $q=\log_{64}y=1/\log_y 64$. The given equations then take the form $p+q=4$ and $\dfrac{1}{p}-\dfrac{1}{q}=1$, whose solutions are $(p_1,q_1)=(3+\sqrt{5},1-\sqrt{5})$ and $(p_2,q_2)=(3-\sqrt{5},1+\sqrt{5})$. Thus $x_1x_2=225^{p_1}225^{p_2}=225^{p_1+p_2}=225^6$, $y_1y_2=64^{q_1+q_2}=$
64^2, and $\log_{30}(x_1y_1x_2y_2)=\log_{30}\left(225^6 64^2\right)=\log_{30}\left(15^{12}2^{12}\right)=\log_{30}30^{12}=12$.

7. (Answer: 428)
Apply the Binomial Expansion to obtain
$$(10^{2002}+1)^{10/7}=10^{2860}+\frac{10}{7}\cdot 10^{3\cdot 286}+\frac{\frac{10}{7}\cdot\frac{3}{7}}{2}\cdot 10^{-4\cdot 286}+\ldots$$
Thus, only the second term affects the requested digits. Since $1/7=.\overline{142857}$ and 6 is a divisor of $3\cdot 286$, conclude that
$$\frac{10}{7}\cdot 10^{3\cdot 286}=1428571\ldots 571.\overline{428571},$$
so the answer is 428.

8. (Answer: 748)
Suppose $a_1 = x_i$ and $a_2 = x_i + h_i$ for $i = 1, 2$, with $x_2 > x_1 > 0$ and $h_1 > h_2 \geq 0$, so

$$a_9 = 34x_1 + 21h_1 = k = 34x_2 + 21h_2.$$

If h_2 were greater than zero, then k would not be the smallest integer for which the equation $34x+21h = k$ has a non-unique solution, since $34x_1+21(h_1-h_2) = 34x_2$ would yield a smaller k. Thus,

$$34x_1 + 21h_1 = 34x_2, \quad \text{that is,} \quad 21h_1 = 34(x_2 - x_1),$$

so h_1 must be a positive multiple of 34, and x_2 and x_1 must differ by a multiple of 21. The smallest possible values of h_1, h_2, x_1, x_2, and a_9 that satisfy these conditions and those of the problem are thus $h_1 = 34$, $h_2 = 0$, $x_1 = 1$, $x_2 = 22$, and $a_9 = 34 \cdot 22 + 21 \cdot 0 = 748$. Note that the sequences

$$1, 35, 36, 71, 107, 178, 285, 463, 748, \ldots$$
$$22, 22, 44, 66, 110, 176, 286, 462, 748, \ldots$$

both have $k = 748$ as their ninth term.

OR

Note that $a_9 = 13a_1+21a_2$, so the requested value of k is the least positive integer k such that $13x + 21y = k$ has more than one solution (x, y) with $0 < x \leq y$ and x and y integers. If k has this property, then there are integers x, y, u and v with $0 < x < u \leq v < y$ and

$$13x + 21y = k = 13u + 21v.$$

Then $21(y - v) = 13(u - x)$ which implies that $u - x$ is divisible by 21. Thus $u - x \geq 21$ and $v \geq u \geq 22$. Now

$$k = 13u + 21v \geq 13 \cdot 22 + 21 \cdot 22 = 748.$$

To demonstrate that $13x + 21y = 748$ has more than one solution, rewrite the equation as $13(x+y)+8y = 57\cdot 13+7$, and conclude that 13 must be a divisor of $(8y-7)$. A few trials reveal that $y = 9$ satisfies this condition. Thus $(43, 9), (43-21, 9+13) = (22, 22)$, and $(43-2\cdot 21, 9+2\cdot 13) = (1, 35)$ are solutions. Note that $(22, 22)$ and $(1, 35)$ yield the previously mentioned sequences, and $(43, 9)$ yields a sequence that satisfies conditions (2) and (3), but not (1).

9. (Answer: 757)
Let the pickets be numbered consecutively $1, 2, 3, \ldots$. Let H, T, and U be the sets of numbers assigned to the pickets painted by Harold, Tanya, and Ulysses, respectively. Then

$$H = \{1, 1+h, 1+2h, 1+3h, \ldots\}$$
$$T = \{2, 2+t, 2+2t, 2+3t, \ldots\}$$
$$U = \{3, 3+u, 3+2u, 3+3u, \ldots\}.$$

Each picket will be painted exactly once if and only if H, T, and U partition the set of positive integers into mutually disjoint subsets. Clearly, h, t and u are each greater than 1. In fact, $h \geq 3$, since if $h = 2$, then 3 is in H. Also $h < 5$, because if $h \geq 5$, then 4 cannot be in H; since 4 cannot be in U, 4 would have to be in T, making $T = \{2, 4, 6, \ldots\}$, which would make $U = \{3, 5, 7, \ldots\}$, since 5 is not in H. But this leaves no possible value for h. Thus $h = 3$ or $h = 4$. When $h = 3$, $H = \{1, 4, 7, \ldots\}$. Now 5 cannot be in U because 7 would be too, but 7 is in H. So 5 is in T, and $T = \{2, 5, 8, \ldots\}$, which means that $U = \{3, 6, 9, \ldots\}$. When $h = 4$, $H = \{1, 5, 9, \ldots\}$. Since 4 cannot be in U, $T = \{2, 4, 6, \ldots\}$, so $U = \{3, 7, 11, \ldots\}$. The two paintable integers are 333 and 424, whose sum is 757.

10. (Answer: 148)
By the Angle-Bisector Theorem, $BD : DC = AB : AC = 12 : 37$, and thus the area of triangle ADC is $37/49$ of the area of triangle ABC. By the Angle-Bisector Theorem, $EG : GF = AE : AF = 3 : 10$, and thus the area of triangle AGF is $10/13$ of the area of triangle AEF. The area of triangle AEF is $3/12$ of the area of triangle AFB, which is in turn $10/37$ of the area of triangle ABC. Since $BC = \sqrt{37^2 - 12^2} = 35$, the area of triangle ABC is 210. It follows that the area of quadrilateral $DCFG$ is

$$\left(\frac{37}{49} - \frac{10}{13} \cdot \frac{3}{12} \cdot \frac{10}{37}\right) 210 = \frac{1110}{7} - \frac{5250}{481} = 158\frac{4}{7} - 10\frac{440}{481},$$

so the requested integer is 148.

11. (Answer: 230)
Place a coordinate system on the cube so that $A = (0,0,0)$, $B = (12,0,0)$, $C = (12,12,0)$, $D = (0,12,0)$, and $P = (12,7,5)$. Point P is the first point where the light hits a face of the cube. Let P_2 be the second point at which the light hits a face, and consider the reflection of the cube in face $BCFG$. Then P_2 is the image of the point at which ray AP next intersects a face of the reflected cube. Continue this process so that the $(k+1)^{\text{st}}$ cube is obtained by reflecting the k^{th} cube in the face containing P_k for $k \geq 2$. Therefore, each intersection of ray AP and a plane with equation $x = 12n$, $y = 12n$, or $z = 12n$, where n is a positive integer, corresponds to a point where the light beam hits a face of the cube. Thus the path will first return to a vertex of the cube when ray AP reaches a point whose coordinates are all multiples of 12. The points on ray AP have coordinates of the form $(12t, 7t, 5t)$, where t is nonnegative, and they will all be multiples of 12 if and only if t is a multiple of 12. This first happens when $t = 12$, which yields the point $(144, 84, 60)$. The requested distance is the same as the distance from this point to A, namely, $12\sqrt{12^2+7^2+5^2} = 12\sqrt{218}$, so $m + n = 230$.

12. (Answer: 275)
Calculate

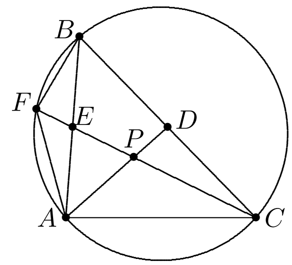
$$F(F(z)) = \frac{\dfrac{z+i}{z-i}+i}{\dfrac{z+i}{z-i}-i} = \frac{z+i+iz+1}{z+i-iz-1} = \frac{1+i}{1-i}\cdot\frac{z+1}{z-1} = i\cdot\frac{z+1}{z-1}$$

and

$$F(F(F(z))) = F\left(i\cdot\frac{z+1}{z-1}\right) = \frac{i\cdot\dfrac{z+1}{z-1}+i}{i\cdot\dfrac{z+1}{z-1}-i} = \frac{z+1+z-1}{z+1-(z-1)} = z,$$

which shows that $z_n = z_{n-3}$ for all $n \geq 3$. In particular, $z_{2002} = z_{2002-667\cdot 3} = z_1 = \frac{\frac{1}{137}+2i}{\frac{1}{137}} = 1 + \frac{2}{\frac{1}{137}}i = 1 + 274i$, and $a + b = 275$.

13. (Answer: 063)



Let P be the intersection of $\overline{AD}$ and $\overline{CE}$. Since angles ABF and ACF intercept the same arc, they are congruent, and therefore triangles ACE and FBE are similar. Thus $EF/12 = 12/27$, yielding $EF = 16/3$. The area of triangle AFB is twice that of triangle AEF, and the ratio of the area of triangle AEF to that of triangle AEP is $\frac{16/3}{9}$, since the medians of a triangle trisect each other. Triangle AEP is isosceles, so the altitude to base $\overline{PE}$ has length $\sqrt{12^2-(9/2)^2} = (1/2)\sqrt{24^2-9^2} = (3/2)\sqrt{8^2-3^2} = (3/2)\sqrt{55}$, and the area of triangle AEP is $(27/4)\sqrt{55}$. Therefore, $[AFB] = 2[AFE] = 2(16/27)\cdot[AEP] = 2(16/27)\cdot(27/4)\sqrt{55} = 8\sqrt{55}$, and $m+n=63$.

14. (Answer: 030)

Let x_1, x_2, x_3, $\ldots$, x_n be the members of $\mathcal{S}$, and let

$$s_j = \frac{x_1+x_2+x_3+\cdots+x_n-x_j}{n-1}\,.$$

It is given that s_j is an integer for any integer j between 1 and n, inclusive. Note that, for any integers i and j between 1 and n, inclusive,

$$s_i - s_j = \frac{x_j-x_i}{n-1},$$

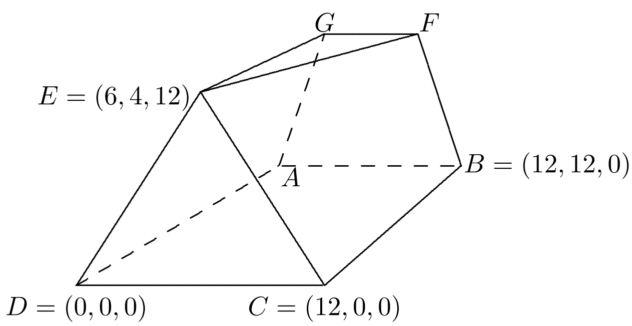
which must be an integer. Also, $x_j = (s_i-s_j)(n-1)+x_i$, and when $x_i = 1$, this implies that each element of $\mathcal{S}$ is 1 more than a multiple of $n-1$. It follows that $(n-1)^2+1 \le 2002$, implying that $n \le 45$. Since $n-1$ is a divisor of $2002-1$, conclude that $n=2$ or $n=4$ or $n=24$ or $n=30$, so n is at most 30. A thirty-element set $\mathcal{S}$ with the requested property is obtained by setting $x_j = 29j-28$ for $1 \le j \le 29$ and $x_{30} = 2002$.

15. (Answer: 163)

Place a coordinate system on the figure so that square $ABCD$ is in the xy-plane, as shown in the diagram. Let $E = (x_1, y_1, 12)$. Because $DE = CE$, it follows that $x_1 = 6$. Because $DE = 14$, it follows that $14^2 = 6^2 + (y_1)^2 + 12^2$, so that $y_1 = 4$. Let $\overline{GK}$ be an altitude of isosceles trapezoid $ABFG$, and notice that the x-coordinates of both G and K are equal to $\frac{1}{2}(AB - GF) = 3$. To find the y-coordinate of G, let $ax + by + cz = d$ be an equation of the plane determined by A, D, and E. Substitute the coordinates of these three points to find that $12b = d$, $0 = d$, and $6a + 4b + 12c = d$, respectively, from which it follows that $b = d = 0$ and $a + 2c = 0$. Thus $G = (3, y_2, z_2)$ lies on the plane $z = 2x$, so $z_2 = 6$. Because $GA = 8$, it follows that $8^2 = 3^2 + (y_2 - 12)^2 + 6^2$, so $y_2 = 12 \pm \sqrt{19}$. Thus

$$EG^2 = (6-3)^2 + (4 - (12 \pm \sqrt{19}))^2 + (12-6)^2 = 128 \pm 16\sqrt{19},$$

and $p + q + r = 163$.



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