

MATHEMATICAL ASSOCIATION OF AMERICA  
AMERICAN MATHEMATICS COMPETITIONS

PRESENTED BY THE AKAMAI FOUNDATION



19<sup>th</sup> Annual

AMERICAN INVITATIONAL  
MATHEMATICS EXAMINATION  
(AIME)

Tuesday, March 27, 2001

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCUTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, calculators are not permitted.
4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY, May 1, 2001.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

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1. Find the sum of all positive two-digit integers that are divisible by each of their digits.
2. A finite set  $\mathcal{S}$  of distinct real numbers has the following properties: the mean of  $\mathcal{S} \cup \{1\}$  is 13 less than the mean of  $\mathcal{S}$ , and the mean of  $\mathcal{S} \cup \{2001\}$  is 27 more than the mean of  $\mathcal{S}$ .
3. Find the sum of all the roots, real and non-real, of the equation  $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$ , given that there are no multiple roots.
4. In triangle  $ABC$ , angles  $A$  and  $B$  measure 60 degrees and 45 degrees, respectively. The bisector of angle  $A$  intersects  $\overline{BC}$  at  $T$ , and  $AT = 24$ . The area of the triangle  $ABC$  can be written in the form  $a + b\sqrt{c}$ , where  $a$ ,  $b$ , and  $c$  are positive integers, and  $c$  is not divisible by the square of any prime. Find  $a + b + c$ .
5. An equilateral triangle is inscribed in the ellipse whose equation is  $x^2 + 4y^2 = 4$ . One vertex of the triangle is  $(0, 1)$ , one altitude is contained in the  $y$ -axis, and the length of each side is  $\sqrt{\frac{m}{n}}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
6. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
7. Triangle  $ABC$  has  $AB = 21$ ,  $AC = 22$ , and  $BC = 20$ . Points  $D$  and  $E$  are located on  $\overline{AB}$  and  $\overline{AC}$ , respectively, such that  $\overline{DE}$  is parallel to  $\overline{BC}$  and contains the center of the inscribed circle of triangle  $ABC$ . Then  $DE = m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
8. Call a positive integer  $N$  a *7-10 double* if the digits of the base-7 representation of  $N$  form a base-10 number that is twice  $N$ . For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?
9. In triangle  $ABC$ ,  $AB = 13$ ,  $BC = 15$  and  $CA = 17$ . Point  $D$  is on  $\overline{AB}$ ,  $E$  is on  $\overline{BC}$ , and  $F$  is on  $\overline{CA}$ . Let  $AD = p \cdot AB$ ,  $BE = q \cdot BC$ , and  $CF = r \cdot CA$ , where  $p$ ,  $q$ , and  $r$  are positive and satisfy  $p + q + r = 2/3$  and  $p^2 + q^2 + r^2 = 2/5$ . The ratio of the area of triangle  $DEF$  to the area of triangle  $ABC$  can be written in the form  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

10. Let  $\mathcal{S}$  be the set of points whose coordinates  $x$ ,  $y$ , and  $z$  are integers that satisfy  $0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ , and  $0 \leq z \leq 4$ . Two distinct points are randomly chosen from  $\mathcal{S}$ . The probability that the midpoint of the segment they determine also belongs to  $\mathcal{S}$  is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
11. In a rectangular array of points, with 5 rows and  $N$  columns, the points are numbered consecutively from left to right beginning with the top row. Thus the top row is numbered 1 through  $N$ , the second row is numbered  $N + 1$  through  $2N$ , and so forth. Five points,  $P_1, P_2, P_3, P_4$ , and  $P_5$ , are selected so that each  $P_i$  is in row  $i$ . Let  $x_i$  be the number associated with  $P_i$ . Now renumber the array consecutively from top to bottom, beginning with the first column. Let  $y_i$  be the number associated with  $P_i$  after renumbering. It is found that  $x_1 = y_2$ ,  $x_2 = y_1$ ,  $x_3 = y_4$ ,  $x_4 = y_5$ , and  $x_5 = y_3$ . Find the smallest possible value of  $N$ .
12. A sphere is inscribed in the tetrahedron whose vertices are  $A = (6, 0, 0)$ ,  $B = (0, 4, 0)$ ,  $C = (0, 0, 2)$ , and  $D = (0, 0, 0)$ . The radius of the sphere is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
13. In a certain circle, the chord of a  $d$ -degree arc is 22 centimeters longer than the chord of a  $3d$ -degree arc, where  $d < 120$ . The length of the chord of a  $3d$ -degree arc is  $-m + \sqrt{n}$  centimeters, where  $m$  and  $n$  are positive integers. Find  $m + n$ .
14. A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?
15. The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. The probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

Your Exam Manager will have a copy of the 2001 AIME Solution Pamphlet.

WRITE TO US :

*Correspondence about the problems and solutions should be addressed to:*

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#### 2001 USAMO

THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 6-hour, essay-type examination. The USAMO will be held on Tuesday, May 1, 2001. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered as indicated below.

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