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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 27, 2001

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 630)

   Let a represent the tens digit and b the units digit of an integer with the required property. Then $10a + b$ must be divisible by both a and b. It follows that b must be divisible by a, and that $10a$ must be divisible by b. The former condition requires that $b = ka$ for some positive integer k, and the latter condition implies that $k = 1$ or $k = 2$ or $k = 5$. Thus the requested two-digit numbers are 11, 22, 33, ..., 99, 12, 24, 36, 48, and 15. Their sum is $11 \cdot 45 + 12 \cdot 10 + 15 = 630$.

2. (Answer: 651)

   Let $\mathcal{S}$ have n elements with mean x. Then

$$\frac{nx+1}{n+1} = x - 13 \qquad \text{and} \qquad \frac{nx+2001}{n+1} = x + 27,$$

   or

$$nx + 1 = (n+1)x - 13(n+1) \qquad \text{and} \qquad nx + 2001 = (n+1)x + 27(n+1).$$

   Subtract the third equation from the fourth to obtain $2000 = 40(n+1)$, from which $n = 49$ follows. Thus $x = 651$.

3. (Answer: 500)

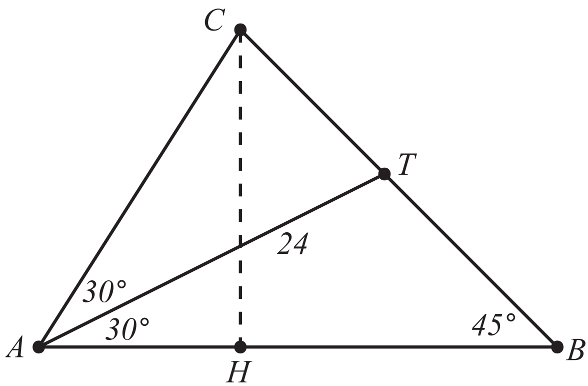
   Apply the binomial theorem to write

$$\begin{aligned} 0 = x^{2001} + \left(\frac{1}{2} - x\right)^{2001} &= x^{2001} - \left(x - \frac{1}{2}\right)^{2001} \\ &= x^{2001} - x^{2001} + 2001 \cdot x^{2000}\left(\frac{1}{2}\right) - \frac{2001 \cdot 2000}{2} x^{1999}\left(\frac{1}{2}\right)^2 + \cdots \\ &= \frac{2001}{2} x^{2000} - 2001 \cdot 250 x^{1999} + \cdots . \end{aligned}$$

   The formula for the sum of the roots yields $2001 \cdot 250 \cdot \frac{2}{2001} = 500$.

4. (Answer: 291)

Note that angles C and ATC each measure $75°$, so $AC = AT = 24$. Draw altitude $\overline{CH}$ of triangle ABC. Then triangle ACH is $30° - 60° - 90°$ and triangle BHC is $45° - 45° - 90°$. Now $AH = 12$ and $BH = CH = 12\sqrt{3}$. The area of triangle ABC is thus $(1/2)12\sqrt{3}(12 + 12\sqrt{3}) = 216 + 72\sqrt{3}$.

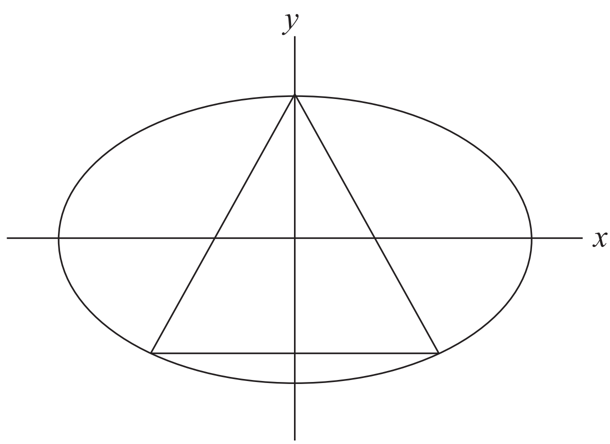


5. (Answer: 937)

Let the other two vertices of the triangle be (x, y) and $(-x, y)$, with $x > 0$. Then the line through $(0, 1)$ and (x, y) forms a 120-degree angle with the positive x-axis, and its slope is $\tan(120°) = -\sqrt{3}$. Therefore, the line's equation is $y = -\sqrt{3}x + 1$. Substituting this into the equation of the ellipse and simplifying yields

$$13x^2 - 8\sqrt{3}x = 0 \qquad \text{or} \qquad x = \frac{8\sqrt{3}}{13}.$$

The triangle has sides of length $2x = (16\sqrt{3})/13 = \sqrt{768/169}$, and $m + n = 937$.



OR

Let the other two vertices of the triangle be (x, y) and $(-x, y)$, with $x > 0$. Equating the squares of the distances from $(0, 1)$ to (x, y) and from $(-x, y)$ to (x, y) yields

$$x^2 + (y-1)^2 = 4x^2, \qquad \text{or} \qquad (y-1)^2 = 3x^2.$$

Substituting from the equation of the ellipse, it follows that $13y^2 - 2y - 11 = 0$. The roots of this quadratic are 1 and -11/13. If $y = 1$, then $x = 0$, so $y = -11/13$. Solving for x yields $x = \sqrt{192/169}$, so that the triangle has sides of length $2x = \sqrt{768/169}$, and $m + n = 937$.

Query: There are two other equilateral triangles with one vertex at $(0, 1)$ that are inscribed in the ellipse $x^2 + 4y^2 = 4$. Can you find the lengths of their sides?

6. (Answer: 079)

Any particular outcome of the four rolls has probability $1/6^4$. Given the values of four rolls, there is exactly one order that satisfies the requirement. It therefore suffices to count all the sets of values that could be produced by four rolls, allowing duplicate values. This is equivalent to counting the number of ways to put four balls into six boxes labeled 1 through 6. By thinking of 4 balls and 5 dividers to separate the six boxes, this can be seen to be $\binom{9}{4} = 126$. The requested probability is thus $126/6^4 = 7/72$, so $m + n = 79$.

OR

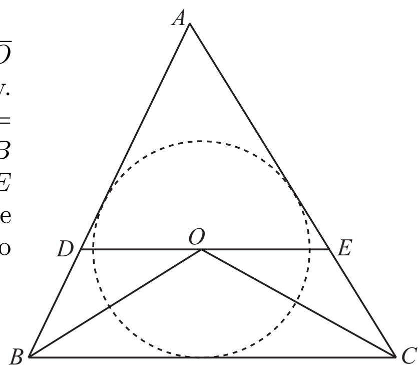
Let a_1, a_2, a_3, and a_4 be the sequence of values rolled, and consider the difference between the last and the first: If $a_4 - a_1 = 0$, then there is 1 possibility for a_2 and a_3, and 6 possibilities for a_1 and a_4. If $a_4 - a_1 = 1$, then there are 3 possibilities for a_2 and a_3, and 5 possibilities for a_1 and a_4. In general, if $a_4 - a_1 = k$, then there are $6 - k$ possibilities for a_1 and a_4, while the number of possibilities for a_2 and a_3 is the same as the number of sets of 2 elements, with repetition allowed, that can be chosen from a set of $k + 1$ elements. This is equal to the number of ways to put 2 balls in $k + 1$ boxes, or $\binom{k+2}{2}$. Thus there are $\sum_{k=0}^{5} \binom{k+2}{2}(6-k) = 126$ sequences of the type requested, so the probability is $126/6^4 = 7/72$, and $m + n = 79$.

OR

Define an *acceptable* sequence to be one in which each element is between 1 and 6 and is at least as large as the preceding element. Let $A(x, n)$ be the number of acceptable sequences of length n beginning with x. Then, for $1 \le x \le 6$, $A(x, 1) = 1$, and $A(x, n)$ is equal to the number of acceptable sequences of length $n - 1$ that begin with a value at least as large as x. That is, $A(x, n) = \sum_{i=x}^{6} A(i, n-1)$. Use this relationship to produce the table shown below. The requested probability is $\frac{56+35+20+10+4+1}{6^4}$ or 7/72.

x	A(x, 1)	A(x, 2)	A(x, 3)	A(x, 4)
1	1	6	21	56
2	1	5	15	35
3	1	4	10	20
4	1	3	6	10
5	1	2	3	4
6	1	1	1	1

7. (Answer: 923)



Let O be the incenter of triangle ABC, so that $\overline{BO}$ and $\overline{CO}$ bisect angles ABC and ACB, respectively. Because $\overline{DE}$ is parallel to $\overline{BC}$, it follows that $\angle DOB = \angle DBO$ and $\angle EOC = \angle ECO$, hence that $DO = DB$ and $EO = EC$. Thus the perimeter if triangle ADE is $AB + AC$. Triangle ADE is similar to triangle ABC, with the ratio of similarity equal to the ratio of perimeters. Therefore

$$\frac{DE}{BC} = \frac{AB + AC}{AB + AC + BC}.$$

Substituting the given values yields $DE = 860/63$, and $m + n = 923$.

OR

Let r be the radius of the inscribed circle, and h be the length of the altitude from A to $\overline{BC}$. Then the area of triangle ABC may be computed in two ways as

$$\frac{1}{2}BC \cdot h = \frac{1}{2}(AB + AC + BC)r, \quad \text{so that} \quad \frac{r}{h} = \frac{BC}{AB + AC + BC}.$$

Triangles ADE and ABC are similar, their ratio of similarity equal to the ratio of any pair of corresponding altitudes. Therefore

$$DE = \frac{h - r}{h} BC = \frac{(AB + AC)BC}{AB + AC + BC}.$$

As above, $m + n = 923$.

8. (Answer: 315)

Suppose that $a_k 7^k + a_{k-1}7^{k-1} + \cdots + a_2 7^2 + a_1 7 + a_0$ is a 7-10 double, with $a_k \neq 0$. In other words, $a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_2 10^2 + a_1 7 + a_0$ is twice as large, so that

$$a_k(10^k - 2\cdot 7^k) + a_{k-1}(10^{k-1} - 2\cdot 7^{k-1}) + \cdots + a_2(10^2 - 2\cdot 7^2) + a_1(10 - 2\cdot 7) + a_0(1-2) = 0.$$

Since the coefficient of a_i in this equation is negative only when $i = 0$ and $i = 1$, and no a_i is negative, it follows that k is at least 2. Because the coefficient of a_i is at least 314 when $i > 2$, and because no a_i exceeds 6 it follows that $k = 2$ and $2a_2 = 4a_1 + a_0$. To obtain the largest possible 7-10 double, first try $a_2 = 6$. Then the equation $12 = 4a_1 + a_0$ has $a_1 = 3$ and $a_0 = 0$ as the solution with the greatest possible value of a_1. The largest 7-10 double is therefore $6 \cdot 49 + 3 \cdot 7 = 315$.

9. (Answer: 061)

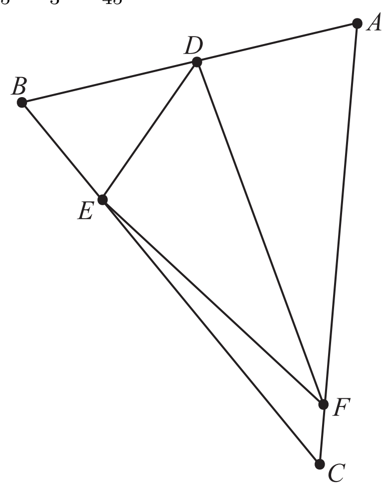
Let $[XYZ]$ denote the area of triangle XYZ. Because p, q, and r are all smaller than 1, it follows that

$$\begin{aligned}
[BDE] &= q(1-p)[ABC], \\
[EFC] &= r(1-q)[ABC], \\
[ADF] &= p(1-r)[ABC], \\
[ABC] &= [DEF] + [BDE] + [EFC] + [ADF] \\
&= [DEF] + ((p+q+r) - (pq+qr+rp))\,[ABC], \text{ and} \\
\frac{[DEF]}{[ABC]} &= 1 + pq + qr + rp - (p+q+r).
\end{aligned}$$

Note that

$$pq + qr + rp = \frac{1}{2}[(p+q+r)^2 - (p^2+q^2+r^2)] = \frac{1}{2}\left(\frac{4}{9} - \frac{2}{5}\right) = \frac{1}{45}.$$

Thus the desired ratio is $1 + \frac{1}{45} - \frac{2}{3} = \frac{16}{45}$ and $m + n = 61$.



10. (Answer: 200)

Because the points of S have integer coordinates, they are called *lattice points*. There are $60 \cdot 59 = 3540$ ways to choose a first lattice point and then a distinct second. In order for their midpoint to be a lattice point, it is necessary and sufficient that corresponding coordinates have the same parity. There are $2^2 + 1^2 = 5$ ways for the first coordinates to have the same parity, including 3 ways in which the coordinates are the same. There are $2^2 + 2^2 = 8$ ways for the second coordinates to have that same parity, including 4 ways in which the coordinates are the same. There are $3^2 + 2^2 = 13$ ways for the third coordinates to have the same parity, including 5 in which the coordinates are the same. It follows that there are $5 \cdot 8 \cdot 13 - 3 \cdot 4 \cdot 5 = 460$ ways to choose two distinct lattice points, so that the midpoint of the resulting segment is also a lattice point. The requested probability is $\frac{460}{3540} = \frac{23}{177}$, so $m + n = 200$.

OR

Because there are $3 \cdot 4 \cdot 5 = 60$ points to choose from, there are $\binom{60}{2} = 1770$ ways to choose the two points. In order that the midpoint of the segment joining the two chosen points also be a lattice point, it is necessary and sufficient that corresponding coordinates have the same parity. Notice that there are

$2 \cdot 2 \cdot 3 = 12$ points whose coordinates are all even,
$1 \cdot 2 \cdot 2 = 4$ points whose coordinates are all odd,
$1 \cdot 2 \cdot 3 = 6$ points whose only odd coordinate is x,
$2 \cdot 2 \cdot 3 = 12$ points whose only odd coordinate is y,
$2 \cdot 2 \cdot 2 = 8$ points whose only odd coordinate is z,
$2 \cdot 2 \cdot 2 = 8$ points whose only even coordinate is x,
$1 \cdot 2 \cdot 2 = 4$ points whose only even coordinate is y, and
$1 \cdot 2 \cdot 3 = 6$ points whose only even coordinate is z.

Thus the desired number of segments is

$$\frac{1}{2}(12 \cdot 11 + 4 \cdot 3 + 6 \cdot 5 + 12 \cdot 11 + 8 \cdot 7 + 8 \cdot 7 + 4 \cdot 3 + 6 \cdot 5) = 230,$$

so that the requested probability is $\frac{230}{1770} = \frac{23}{177}$.

11. (Answer: 149)

Suppose that P_i is in row i and column c_i. It follows that

$$x_1 = c_1, x_2 = N + c_2, x_3 = 2N + c_3, x_4 = 3N + c_4, x_5 = 4N + c_5$$

and

$$y_1 = 5c_1 - 4, y_2 = 5c_2 - 3, y_3 = 5c_3 - 2, y_4 = 5c_4 - 1, y_5 = 5c_5.$$

The P_i have been chosen so that

$$\begin{aligned} c_1 &= 5c_2 - 3 \\ N + c_2 &= 5c_1 - 4 \\ 2N + c_3 &= 5c_4 - 1 \\ 3N + c_4 &= 5c_5 \\ 4N + c_5 &= 5c_3 - 2 \end{aligned}$$

Use the first two equations to eliminate c_1, obtaining $24c_2 = N + 19$. Thus $N = 24k + 5$, where $k = c_2 - 1$. Next use the remaining equations to eliminate c_3 and c_4, obtaining $124c_5 = 89N + 7$. Substitute for N to find that $124c_5 = 2136k + 452$, and hence $31c_5 = 534k + 113 = 31(17k + 3) + 7k + 20$. In other words, $7k + 20 = 31m$ for some positive integer m. Now $7k = 31m - 20 = 7(4m - 2) + 3m - 6$. Since 7 must divide $3m - 6$, the minimum value for m is 2, and the smallest possible value of k is therefore 6, which leads to $N = 24 \cdot 6 + 5 = 149$. It is not difficult to check that $c_2 = 7$, $c_1 = 32$, $c_5 = 107$, $c_4 = 5c_5 - 3N = 88$, and $c_3 = 5c_4 - 1 - 2N = 141$ define an acceptable placement of points P_i. The numbers associated with the points are $x_1 = 32$, $x_2 = 156$, $x_3 = 439$, $x_4 = 535$, and $x_5 = 703$.

Note: Modular arithmetic can be used to simplify this solution.

12. (Answer: 005)

Let r be the radius of the inscribed sphere. Because $ABCD$ can be dissected into four tetrahedra, all of which meet at the incenter, have a height of length r, and have a face of the large tetrahedron as a base, it follows that r times the surface area of $ABCD$ equals three times the volume of $ABCD$. To find the area $[ABC]$ of triangular face ABC, first calculate $AB = \sqrt{52}$, $BC = \sqrt{20}$, and $CA = \sqrt{40}$. Then apply the Law of Cosines to find that $\cos\angle CAB = 9/\sqrt{130}$. It follows that $\sin\angle CAB = 7/\sqrt{130}$, so that $[ABC] = \frac{1}{2} \cdot AB \cdot AC \cdot \sin\angle CAB = 14$. The surface area of $ABCD$ is

$$[ABC] + [ABD] + [ACD] + [BCD] = 14 + 12 + 6 + 4 = 36.$$

The volume of tetrahedron $ABCD$ is $\frac{1}{3} \cdot 2 \cdot \frac{1}{2} \cdot 4 \cdot 6 = 8$. Thus $r = 24/36 = 2/3$ and $m + n = 5$.

OR

Because the sphere is tangent to the xy-plane, the yz-plane, and the xz-plane, its center is (r, r, r), where r is the radius of the sphere. An equation for the plane of triangle ABC is $2x+3y+6z=12$, so the sphere is tangent to this plane at $(r+2t, r+3t, r+6t)$, for some positive number t. Thus $2(r+2t)+3(r+3t)+6(r+6t)=12$ and $(2t)^2+(3t)^2+(6t)^2=r^2$, from which follow $11r+49t=12$ and $7t=r$, respectively. Combine these equations to discover that $r=2/3$ and $m+n=5$.

OR

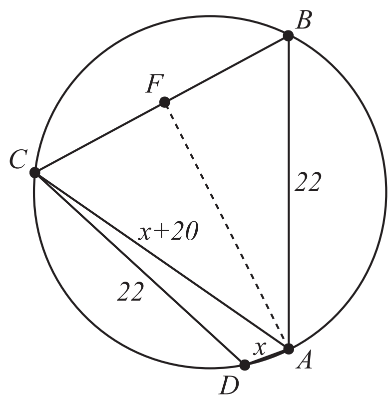
An equation of the plane of triangle ABC is $2x+3y+6z=12$. The distance from the plane to (r, r, r) is

$$\frac{|2r+3r+6r-12|}{\sqrt{2^2+3^2+6^2}}.$$

This leads to $\frac{|11r-12|}{7}=r$, which is satisfied by $r=3$ and $r=2/3$. Since $(3,3,3)$ is outside the tetrahedron, $r=2/3$ and $m+n=5$.

Query: The sphere determined by $r=3$ is outside the tetrahedron and tangent to the planes containing its faces. Can you find the radii of the other three spheres with this property?

13. (Answer: 174)



In the figure, points A, B, C, and D are concyclic, the degree sizes of arcs AB, BC, and CD are all d, and $AB=BC=CD=22$. Note that $\overline{AD}$ is the chord of a $3d$-degree arc. Let $AD=x$. Then $AC=x+20$, because $\overline{AC}$ is the chord of a $2d$-degree arc. In isosceles trapezoid $ABCD$, draw the altitude $\overline{AF}$ from A to $\overline{BC}$, and notice that F divides BC into $BF=11-\frac{x}{2}$ and $CF=11+\frac{x}{2}$. Because the right triangles AFC and AFB share the leg $\overline{AF}$, it follows that

$$(x+20)^2-(11+\frac{x}{2})^2=22^2-(11-\frac{x}{2})^2,$$

which simplifies to $x^2+18x-84=0$. Thus $x=-9+\sqrt{165}$ and $m+n=174$.

OR

Noting that $ABCD$ is a cyclic isosceles trapezoid, apply Ptolemy's Theorem to obtain $AB \cdot BD = BC \cdot AD + CD \cdot AB$, or $(x+20)^2=22x+22^2$. Solve the equation to find that $x=-9+\sqrt{165}$.

Query: If the restriction $d<120$ were removed, then the problem would have an additional solution. Can you find it?

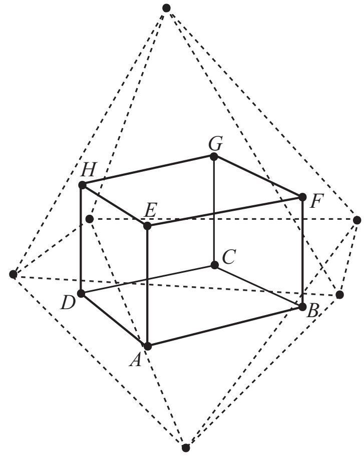
14. (Answer: 351)

The first condition implies that at most ten houses get mail in one day, while the second condition implies that at least six houses get mail. If six houses get mail, they must be separated from each other by a total of at least five houses that do not get mail. The other eight houses that do not get mail must be distributed in the seven spaces on the sides of the six houses that do get mail. This can be done in 7 ways: put two at each end of the street and distribute the other four in $\binom{5}{4} = 5$ ways, or put one in each of the seven spaces and an extra one at one end of the street or the other. If seven houses get mail, they create eight spaces, six of which must contain at least one house that does not get mail. The remaining six houses that do not get mail can be distributed among these eight spaces in 113 ways: six of the eight spaces can be selected to receive a single house in $\binom{8}{6} = 28$ ways; two houses can be placed at each end of the street and two intermediate spaces be selected in $\binom{6}{2} = 15$ ways; and two houses can be placed at one end of the street and four spaces selected for a single house in $2\binom{7}{4} = 70$ ways. Similar reasoning shows that there are $\binom{9}{1} + 1 + 2\binom{8}{2} = 183$ patterns when eight houses get mail, and $2 + \binom{10}{2} = 47$ patterns when nine houses get mail. When ten houses get mail, there is only one pattern, and thus the total number of patterns is $7 + 113 + 183 + 47 + 1 = 351$.

OR

Consider n-digit strings of zeros and ones, which represent no mail and mail, respectively. Such a sequence is called *acceptable* if it contains no occurrences of 11 or 000. Let f_n be the number of acceptable n-digit strings, let a_n be the number of acceptable n-digit strings in which 00 follows the leftmost 1, and let b_n be the number of acceptable n-digit strings in which 01 follows the leftmost 1. Notice that $f_n = a_n + b_n$ for $n \geq 5$. Deleting the leftmost occurrence of 100 shows that $a_n = f_{n-3}$, and deleting 10 from the leftmost occurrence of 101 shows that $b_n = f_{n-2}$. It follows that $f_n = f_{n-2} + f_{n-3}$ for $n \geq 5$. It is straightforward to verify the values of $f_1 = 2$, $f_2 = 3$, $f_3 = 4$, and $f_5 = 7$. Then the recursion can be used to find that $f_{19} = 351$.

15. (Answer: 085)



It is helpful to consider the cube $ABCDEFGH$ shown in the figure. The vertices of the cube represent the faces of the dotted octahedron, and the edges of the cube represent adjacent octahedral faces. Each assignment of the numbers 1,2,3,4,5,6,7, and 8 to the faces of the octahedron corresponds to a permutation of $ABCDEFGH$, and thus to an octagonal circuit of these vertices. The cube has 16 diagonal segments that join nonadjacent vertices. In effect, the problem asks one to count octagonal circuits that can be formed by eight of these diagonals. Six of the diagonals are edges of tetrahedron $ACFH$, six are edges of tetrahedron $DBEG$, and four are *long*, joining a vertex of one tetrahedron to the diagonally opposite point from the other. Notice that each vertex belongs to exactly one long diagonal. It follows that an octagon cannot have two successive long diagonals. Also notice that an octagonal path can jump from one tetrahedron to the other only along one of the long diagonals. it follows that an octagon must contain either 2 long diagonals separated by 3 tetrahedron edges or 4 long diagonals alternating with tetrahedron edges. To form an octagon that contains four long diagonals, choose two opposite edges from tetrahedron $ACFH$ and two opposite edges from tetrahedron $DBEG$. For each of the three ways to choose a pair of opposite edges from tetrahedron $ACFH$, there are two possible ways to choose a pair of opposite edges from tetrahedron $DBEG$. There are 6 distinct octagons of this type and $8 \cdot 2$ ways to describe each of them, making 96 permutations. To form an octagon that contains exactly two of the long diagonals, choose a three-edge path along tetrahedron $ACFH$, which can e done in $4! = 24$ ways. Then choose a three-edge path along tetrahedron $DBEG$ which, because it must start and finish at specified vertices, can be done in only 2 ways. Since this counting method treats each path as different from its reverse, there are $8 \cdot 24 \cdot 2 = 384$ permutations of this type. In all, there are $96 + 384 = 480$ permutations that correspond to octagonal circuits formed exclusively from cube diagonals.
The probability of randomly choosing such a permutation is $\frac{480}{8!} = \frac{1}{84}$, $m + n = 85$.
Note: The cube is called the emphdual of the octahedron.

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